Martingale representation property in progressively enlarged filtrations

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Abstract. Consider $\mathbb{G}$ the progressive enlargement of a filtration $\mathbb{F}$ with a random time τ. Assuming that, in $\mathbb{F}$, the martingale representation property holds, we examine conditions under which the martingale representation property holds also in $\mathbb{G}$. It is noted that the classical results on this subject are no more sufficient to deal with all examples coming from credit risk modeling. In this paper, we introduce a new methodology which extends the various classical results and applies on recent examples.

Key words. Progressive enlargement of filtration, random time, martingale representation property, honest time, immersion condition, change of probability measures, $(\mathcal{H}')$ hypothesis, credit risk modelling, Cox model.

MSC class. 60G07, 60G44.

1   Introduction

Our paper examines conditions under which a progressively enlarged filtration possesses the martingale representation property (in short $\mathfrak{M}$rp). This study is motivated by the credit risk modeling, especially by the models with multiple defaults (see [10, 22, 23]) where one needs to know the market behavior after the occurrences of the defaults to make running market activities (pricing, hedging, optimization, etc). We seek general methods to meet the wide range of models. Results exist under the classical assumptions (Jacod criterion, honest time, immersion, etc) (see [1, 5, 7, 17, 21, 26]). They are, however, not enough to cover the new models introduced in a recent work [19]. This paper completes the gap.

(The most notations used in this paper are usual ones. Otherwise, see Section 1.5 for detailed definitions)

1.1 $(\mathcal{H}')$ hypothesis

Let us recall some basic facts on the progressive enlargement of filtration.

We consider a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions. Let τ be an $\mathcal{A}$-measurable random variable taking values in $[0, \infty]$ and $\mathbb{G}$ be the progressive enlargement of the filtration $\mathbb{F}$ by the random time $\tau : \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ with $\mathcal{G}_t = \cap_{s > t}(\mathcal{F}_s \vee$

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$\sigma(\tau \wedge s))$. We augment $\mathbb{G}$ by the $(\mathbb{Q}, \mathcal{G}_\infty)$ negligible sets and we denote again by $\mathbb{G}$ the augmented filtration.

The $(\mathbb{Q}, \mathbb{F})$ optional projection of $\mathbb{1}_{[0,\tau)}$, denoted by Z, is a bounded and positive $(\mathbb{Q}, \mathbb{F})$ supermartingale. We denote by $Z = M - A$ its Doob-Meyer's decomposition, where M is a $(\mathbb{Q}, \mathbb{F})$ martingale and A is a càdlàg $(\mathbb{Q}, \mathbb{F})$-predictable increasing process with $A_0 = 0$. We note that A coincides with the $(\mathbb{Q}, \mathbb{F})$ predictable dual projection of $\mathbb{1}_{\{\tau > 0\}} \mathbb{1}_{[\tau, \infty)}$. We consider also the $(\mathbb{Q}, \mathbb{F})$ optional dual projection of $\mathbb{1}_{\{\tau > 0\}} \mathbb{1}_{[\tau, \infty)}$, that we denote by $\hat{A}$. Let $H := \mathbb{1}_{[\tau, \infty)}$. According to [21, Remarques(4.5) 3)], the process

$$L_t = \mathbb{1}_{\{\tau > 0\}} H_t - \int_0^{t \wedge \tau} \frac{dA_s}{Z_{s-}}, \ t \geq 0. \tag{1}$$

is a $(\mathbb{Q}, \mathbb{G})$ local martingale.

The fundamental question in the enlargement of filtrations setting is to know what becomes a $(\mathbb{Q}, \mathbb{F})$ martingale in the filtration $\mathbb{G}$. Let $\mathcal{M}_{loc}(\mathbb{Q}, \mathbb{F})$ be the space of $(\mathbb{Q}, \mathbb{F})$ local martingales. We introduce the following $(\mathcal{H}')$-hypothesis :

Assumption 1.1 $(\mathcal{H}')$ **Hypothesis** *We assume that there exists an application Γ from $\mathcal{M}_{loc}(\mathbb{Q}, \mathbb{F})$ into the space of càdlàg $\mathbb{G}$-predictable processes with finite variation, such that, for any $X \in \mathcal{M}_{loc}(\mathbb{Q}, \mathbb{F})$, $\Gamma(X)_0 = 0$ and $\widetilde{X} := X - \Gamma(X)$ is a $(\mathbb{Q}, \mathbb{G})$ local martingale. The operator Γ will be called the drift operator.*

Note that, when a $(\mathbb{Q}, \mathbb{F})$ local martingale X is a $(\mathbb{Q}, \mathbb{G})$-semimartingale, X is a special semimartingale. Consequently, the drift operator $\Gamma(X)$ is well defined. Many facts are known about the drift operator Γ (see [21]). In particular, for any random time τ, the drift operator Γ on $(0, \tau]$ takes the universal form :

$$\mathbb{1}_{(0,\tau]} \cdot \Gamma(X) = \mathbb{1}_{(0,\tau]} \frac{1}{Z_-} \cdot \langle M + \hat{A} - A, X \rangle \tag{2}$$

where $\cdot$ denotes a stochastic integral and $\langle \cdot \rangle$ denotes the predictable bracket of local martingales computed in the filtration $\mathbb{F}$. The situation for the drift operator $\Gamma(X)$ on the time interval (τ, ∞) is more complicated. There do not exist general results. The only known example until recently was the case of a honest time τ for which the drift operator on (τ, ∞) is given by :

$$\mathbb{1}_{(\tau,\infty)} \cdot \Gamma(X) = \mathbb{1}_{(\tau,\infty)} \frac{1}{1 - Z_-} \cdot \langle M + \hat{A} - A, X \rangle \tag{3}$$

(see [1], [24]). Recent researchs (cf. [7, 9, 16, 18, 19]) show that $\mathbb{1}_{(\tau,\infty)} \cdot \Gamma(X)$ can have various different forms.

1.2 The notion of local solutions

The martingale representation property $\mathfrak{Mrp}$ in $\mathbb{G}$ has been studied under divers assumptions ([1, 5, 7, 17, 21, 26]). Each time, one makes use of a different method.

Our approach to $\mathfrak{Mrp}$ in $\mathbb{G}$ is based on the thesis [29]. Since the seminal Jacod's paper [15], the method of probability changes is known to be a good tool for solving the enlargement of filtration problem. However, in many situations (even the simplest ones) this method does not work. For example, let X be a Brownian motion issued from 0 and consider the initial enlargement $\mathbb{F} \vee \sigma(X_1)$. By a direct computation, we see that X is a $\mathbb{F} \vee \sigma(X_1)$ semimartingale. But this result can not be obtained as a consequence of Jacod's criterion. We note that Jacod's criterion is actually applicable on the time intervals $[0, 1 - \delta]$ for any $0 < \delta < 1$, but not beyond.

The problem raised in the above observation had been studied in [29]. It comes to the conclusion that, to solve a problem of enlargement of filtration with the method of probability changes, one should distinguish two different aspects and proceed in several steps. The first one consists to see if, for each $t \in \mathbb{R}_+$, there exists locally a no empty random interval $(t, T_t]$ on which the method of probability changes applies to yield a solution. The second one concerns how to integrate the local solutions into a global one on the union set $\cup_{t \in \mathbb{R}_+}(t, T_t]$. After these two steps, one obtains the complementary set $(\cup_{t \in \mathbb{R}_+}(t, T_t])^c$ which represents where the method of probability changes no longer works and the problem should be traited differently.

The local solution method fits very well the classical examples of enlargement of filtrations, where the complementary set $(\cup_{t \in \mathbb{R}_+}(t, T_t])^c$ usually reduces into a finite set. It fits as well the study of $\mathfrak{Mrp}$ in $\mathbb{G}$, as we will see below. In addition, the classical results on $\mathfrak{Mrp}$ in $\mathbb{G}$ are consequences of the local solution method, as shown in Section 6. In Section 7 we give a proof of $\mathfrak{Mrp}$ for the models in [19].

1.3   Method preview

We suppose $(\mathcal{H}')$ hypothesis and property $\mathfrak{Mrp}$ hold in $\mathbb{F}$ with a driving process W. We begin with the search of local $\mathfrak{Mrp}$ on the time interval $[0, \tau]$. The general situation on $[0, \tau]$ is described in a lemma from [21] :

Lemma 1.1 *Let N be a bounded $(\mathbb{Q}, \mathbb{G})$ local martingale such that $N_0 = 0, \Delta_\tau N = 0$. Suppose that N is orthogonal to every $\widetilde{X} = X - \Gamma(X)$, where X runs over the family of all $(\mathbb{Q}, \mathbb{F})$ local martingales. Then, $N \equiv 0$ on $[0, \tau]$*

This result suggests that $\widetilde{W}$ should be a driving process for $\mathfrak{Mrp}$ in $\mathbb{G}$. On the other hand, the following formula (see [4]) satisfied by any bounded $(\mathbb{Q}, \mathbb{G})$ martingales N stopped at τ :

$$N_t^\tau = \frac{\mathbb{Q}[N_\tau \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t]}{Z_t}(1 - H_t) + N_\tau H_t, \ 0 \leq t < \infty \tag{4}$$

links the $\mathfrak{Mrp}$ in $\mathbb{F}$ to $\mathfrak{Mrp}$ in $\mathbb{G}$. See Section 3 for details.

Now consider $\mathfrak{Mrp}$ on (τ, ∞). To illustrate our idea, let us suppose that the drift operator $\Gamma(X)$ on (τ, ∞) takes the form

$$\int_0^t \mathbb{1}_{\{\tau < s\}} d\Gamma_s(X) = \int_0^t \mathbb{1}_{\{\tau < s\}} \beta_s d\langle Y, X\rangle_s,$$

where Y is a $(\mathbb{Q}, \mathbb{G})$ local martingale and β is some $\mathbb{G}$-predictable process. This form reminds the Girsanov formula. We take now two $\mathbb{G}$-stopping times S, T such that $\tau \leq S \leq T$ and we consider the stochastic differential equation

$$d\eta_t = -\eta_{t-} \mathbb{1}_{\{S < t \leq T\}} \beta_t dY_t$$

Suppose that the solution η is a strictly positive $(\mathbb{Q}, \mathbb{G})$ uniformly integrable martingale. We set $\mathbb{Q}' = \eta_T \cdot \mathbb{Q}$. By Girsanov theorem, any $(\mathbb{Q}, \mathbb{F})$ local martingale X will be a $(\mathbb{Q}', \mathbb{G})$ local martingale on the time interval $(S, T]$. This is a situation very similar to the immersion condition.

Inspired by the work in [26], we consider the $(\mathbb{Q}', \mathbb{G})$ martingale of the form

$$\mathbb{Q}'[g(\tau)\zeta | \mathcal{G}_t], t \geq \tau,$$

(where g is a bounded Borel function and $\zeta \in \mathcal{F}_\infty$ is bounded). Under the condition that $\zeta \in \mathcal{F}_T$ and $\mathcal{F}_T \subset \mathcal{F}_\infty$ (in particular when T is a $\mathbb{F}$ stopping time), we can prove that the above $(\mathbb{Q}', \mathbb{G})$ martingale has a stochastic integral representation with respect to W on the time interval $(S, T]$. This means that $\mathfrak{Mrp}$ holds under $\mathbb{Q}'$ in a filtration $\mathbb{G}^{(S,T]}$ (see Definition 4.1), where the filtration $\mathbb{G}^{(S,T]}$ is constructed in such a way that it coincides with $\mathbb{G}$ on the time interval (S, T), whilst its terminal term $\mathcal{G}_\infty^{(S,T]}$ is a

subset of $\sigma(\tau) \vee \mathcal{F}_T$. (Note that in general, we do not have $\mathcal{G}_T = \sigma(\tau) \vee \mathcal{F}_T$, which is one of the main technical difficulties to establish $\mathfrak{Mrp}$ in $\mathbb{G}$.) Noting that $\mathfrak{Mrp}$ is invariant under change of probability measure, we conclude finally a $\mathfrak{Mrp}$ under $\mathbb{Q}$ in the filtration $\mathbb{G}^{(S,T]}$, considered as a local solution of $\mathfrak{Mrp}$ in $\mathbb{G}$.

The key point in the above argument is the existence of the probability $\mathbb{Q}'$. Such a probability will be called an $\mathcal{SH}$ measure in Section 4.2.

After obtaining the local solutions, their integration can be done using Lemma 2.3 below. Technical problems rise, however, when we pass from the filtrations $\mathbb{G}^{(S,T]}$ into $\mathbb{G}$, caused by eventual jumps at T of $(\mathbb{Q}, \mathbb{G})$ local martingales, which are not $\sigma(\tau) \vee \mathcal{F}_T$ measurable. See Section 4 for details.

1.4 The equality between $\mathcal{G}_\tau$ and $\mathcal{G}_{\tau-}$

It is to note that $\mathfrak{Mrp}$ property in the progressively enlarged filtration is closely linked with the σ-algebra equality $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau = \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$. This equality will also be treated with the local solution method. Our result here is a complement to the classical results such as [2]. See Section 5.

1.5 Notations and definitions

In this paper, calling a number a positive means that $a \geq 0$ and calling a function $f(t), t \in \mathbb{R}$, an increasing function means $f(s) \leq f(t)$ for $s \leq t$.

When we say that a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual condition under a probability $\mathbb{P}$, we consider only the $(\mathbb{P}, \mathcal{F}_\infty)$-negligible sets in the usual condition.

Relations between random variables is to be understood to be almost sure relations. For a random variable X and a σ-algebra $\mathcal{F}$, the expression $X \in \mathcal{F}$ means that X is $\mathcal{F}$-measurable.

We will compute expectations with respect to different probability measures $\mathbb{P}$. To simplify the notation, we will denote the expectation by $\mathbb{P}[\cdot]$ instead of $\mathbb{E}_\mathbb{P}[\cdot]$, and the conditional expectation by $\mathbb{P}[\cdot|\mathcal{F}_t]$ instead of $\mathbb{E}_\mathbb{P}[\cdot|\mathcal{F}_t]$.

By random time we mean a positive random variable. For any random time υ, $\mathcal{F}_\upsilon$ (resp. $\mathcal{F}_{\upsilon-}$) denotes the σ-algebra generated by the random variables $U_\upsilon \mathbb{1}_{\{\upsilon < \infty\}} + \xi \mathbb{1}_{\{\upsilon = \infty\}}$ where U runs over the family of the $\mathbb{F}$-optional processes (respectively $\mathbb{F}$-predictable processes) and $\xi \in \mathcal{F}_\infty$

Let D be a subset of Ω and $\mathcal{T}$ be a σ-algebra on Ω. We denote by $D \cap \mathcal{T}$ the family of all subsets $D \cap A$ with A running through $\mathcal{T}$. If D itself is an element in $\mathcal{T}$, $D \cap \mathcal{T}$ coincides with $\{A \in \mathcal{T} : A \subset D\}$. Although $D \cap \mathcal{T}$ is a σ-algebra on D, we will use it mainly as a family of subsets of Ω. We use the symbol "+" to present the union of two disjoint subsets. For two disjoint sets D_1, D_2 in Ω, and two families $\mathcal{T}_1, \mathcal{T}_2$ of sets in Ω, we denote by $D_1 \cap \mathcal{T}_1 + D_2 \cap \mathcal{T}_2$ the family of sets $D_1 \cap B_1 + D_2 \cap B_2$ where $B_1 \in \mathcal{T}_1, B_2 \in \mathcal{T}_2$. For a probability $\mathbb{P}$, we say $D \cap \mathcal{T}_1 = D \cap \mathcal{T}_2$ under $\mathbb{P}$, if, for any $A \in \mathcal{T}_1$, there exists a $B \in \mathcal{T}_2$ such that $(D \cap A)\Delta(D \cap B)$ is $\mathbb{P}$-negligible, and vice versa.

For a càdlàg process X, the jump $\Delta_s X$ is defined so that $\Delta_0 X = X_0$ and $\Delta_\infty X = 0$.

Let X be a semimartingale and J be a predictable process, integrable with respect to the semimartingale X (here we adopt the definition of the integrability in [13]). The stochastic integral $(\int_0^t J_s dX_s)_{t \geq 0}$ will be denoted by $J \boldsymbol{.} X$, and $J \boldsymbol{.} X_0$ will be by definition zero.

Two local martingales X', X'' are said to be orthogonal, if the product $X'X''$ is again a local martingale.

2 Some basic results

In this section, we recall some basic facts on the notion of stochastic integrals and on the martingale representation property.

2.1 Stochastic integrals in different filtrations under different probability measures

A technical problem in this paper is that a stochastic integral defined in a filtration under some probability measure will be also considered in another filtration under another probability measure. Lemma 2.1 given below states conditions which ensure that the variously defined stochastic integrals coincide (see [21, Corollaire (1.18)]).

Assumption 2.1 *Let $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$ and $\hat{\mathbb{G}} = (\hat{\mathcal{G}}_t)_{t \geq 0}$ be two right-continuous filtrations on a measurable space $(\Omega, \mathcal{A})$. Let $\hat{\mathbb{P}}$ and $\hat{\mathbb{Q}}$ be two probability measures on $\mathcal{A}$. Let X be a càdlàg process and $0 \leq S \leq T$ be two given random times. Suppose that*

1. *S, T are $\hat{\mathbb{F}}$-stopping times and $\hat{\mathbb{G}}$-stopping times.*
2. *X is a $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ semimartingale and a $(\hat{\mathbb{Q}}, \hat{\mathbb{G}})$ semimartingale.*
3. *$(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ (resp. $(\hat{\mathbb{Q}}, \hat{\mathbb{G}})$) satisfies the usual condition.*
4. *The probability $\hat{\mathbb{Q}}$ is equivalent to $\hat{\mathbb{P}}$ on $\hat{\mathcal{F}}_\infty \vee \hat{\mathcal{G}}_\infty$*
5. *for any $\hat{\mathbb{F}}$-predictable process J, $J\mathbb{1}_{(S,T]}$ is a $\hat{\mathbb{G}}$-predictable process.*

Lemma 2.1 *Suppose Assumption 2.1. Let J be an $\hat{\mathbb{F}}$-predictable process. Suppose that $J\mathbb{1}_{(S,T]}$ is integrable with respect to X in the sense of $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ semimartingale, as well as, in the sense of $(\hat{\mathbb{Q}}, \hat{\mathbb{G}})$ semimartingale. Then, the stochastic integral $J\mathbb{1}_{(S,T]} \cdot X$ defined in the two senses gives the same process.*

Proof. If the process J is elementary, the conclusion of the lemma is obviously true. By dominated convergence theorem of stochastic integral (cf. [13, Theorem 9.30]) and by monotone class theorem, the lemma is also true for bounded J. The result is then extended to a general J using the fact that, in the two filtrations,

$$J\mathbb{1}_{(S,T]} \cdot X_t = \lim_{n \to \infty} ((-n) \vee (J \wedge n)\mathbb{1}_{(S,T]}) \cdot X_t, t \geq 0,$$

(convergence in probability). ∎

2.2 Martingale representation property

In this section, we give the definition of the martingale representation property that we adopt in this paper, and we recall some related results. Let $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ be a filtered probability space satisfying the usual conditions. We introduce the space $\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ (resp. the space $\mathcal{M}_0^\infty(\hat{\mathbb{P}}, \hat{\mathbb{F}})$) of all $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingales M (resp. of all bounded martingales M) null at the origin $M_0 = 0$. Let $\mathcal{H}_0^p = \mathcal{H}_0^p(\hat{\mathbb{P}}, \hat{\mathbb{F}}), p \geq 1$, be the normed space of the $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingales M such that $M_0 = 0$ and $\mathbb{E}[(\sqrt{[M]_\infty})^p] < \infty$ (cf. [13, Chapter 10] and [14]).

Consider a d-dimensional $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingale W. We denote by $\mathcal{I}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ the family of all d-dimensional $\hat{\mathbb{F}}$-predictable processes $J = (J_i)_{1 \leq i \leq d}$ such that the component J_i is integrable with

respect to W_i, for $1 \le i \le d$, under the probability $\hat{\mathbb{P}}$ in the sense of local martingale. We consider the family of local martingales

$$\mathfrak{W} = \{J \centerdot W = \sum_{i=1}^{d} J_i \centerdot W_i : J \in \mathcal{I}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)\}$$

and we denote by $\mathcal{M}_0(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ the closure of $\mathfrak{W} \cap \mathcal{H}_0^1$ in the space $\mathcal{H}_0^1$ (cf. [14]). We denote by $\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ the family of local martingales which are locally in $\mathcal{M}_0(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$.

We introduce the following condition :

Assumption 2.2 *Ellipticity There exists a positive constant C such that*

$$\sum_{i=1}^{d} \int_0^\infty (J_{i,s})^2 d[W_i]_s \le C \sum_{k,j=1}^{d} \int_0^\infty J_{k,s} J_{j,s} d[W_k, W_j]_s, \ \hat{\mathbb{P}} - a.s.$$

for any d-dimensional measurable processes J such that $\int_0^\infty |J_{k,s} J_{j,s}| |d[W_k, W_j]_s| < \infty$ for all $1 \le k, j \le d$.

Remark 2.1 We can check that, under the ellipticity assumption, the subspace $\mathfrak{W} \cap \mathcal{H}_0^1$ is closed in the space $\mathcal{H}_0^1$ and we have

$$\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W) = \{J \centerdot W = \sum_{i=1}^{d} J_i \centerdot W_i : J \in \mathcal{I}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)\} \tag{5}$$

In [14] a general notion of stochastic integral with respect to a finite family of local martingales is defined. We underline that, in this paper, the stochastic integral $J \centerdot W$ with respect to a finite dimensional local martingale W is to be understood as component by component. The above formula (5) means (cf. [14, Remarques (4.36)]) that, under the ellipticity assumption, our definition coincides with that of [14]. Note also that the ellipticity assumption is satisfied if the W_i's are strongly orthogonal : $[W_k, W_j] = 0$ whenever $k \ne j$.

Martingale representation property. Let W be a d-dimensional càdlàg $\hat{\mathbb{F}}$-adapted process. We say that the martingale representation property holds in the filtration $\hat{\mathbb{F}}$ under the probability $\hat{\mathbb{P}}$ with respect to the driving process W, if W is a d-dimensional $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingale, and if

$$\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W) = \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$$

The martingale representation property will be denoted by $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$, or simply by $\mathfrak{Mrp}$. We note that, if $\mathfrak{Mrp}$ holds with the ellipticity assumption, for any $X \in \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$, X is a stochastic integral with respect to W : $X = J \centerdot W$ for a $J \in \mathcal{I}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$.

We introduce the operator [1] $\dagger$: For ζ a $\hat{\mathcal{F}}_\infty$-measurable $\hat{\mathbb{P}}$-integrable random variable, we denote by $\dagger \zeta$ the martingale

$$\dagger \zeta_t = \hat{\mathbb{P}}[\zeta | \hat{\mathcal{F}}_t] - \hat{\mathbb{P}}[\zeta | \hat{\mathcal{F}}_0], t \ge 0.$$

Lemma 2.2 *Suppose that W is a d-dimensional $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingale. Let $\mathcal{C}$ be a π-class contained in $\hat{\mathcal{F}}_\infty$ such that $\sigma(\mathcal{C}) = \hat{\mathcal{F}}_\infty$. If, for any $A \in \mathcal{C}$, the $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$-martingale $\dagger \mathbb{1}_A$ is an element in $\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$, then $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ holds.*

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1. Note that the operator $\dagger$ depends on $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$. The context in which $\dagger$ is used will help to avoid the ambiguity.

Proof. We know that, if $\mathcal{M}_0^\infty(\hat{\mathbb{P}}, \hat{\mathbb{F}}) \subset \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$, then the space $\mathcal{H}_0^1$ is contained in $\mathcal{M}_0(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ and consequently $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ holds (cf. [13]). Let Π be the family of bounded $\hat{\mathcal{F}}_\infty$-measurable random variables ζ such that the $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$-martingale $\dagger \zeta$ belongs to $\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$. Applying the monotone class theorem, we see that the space Π contains all bounded $\sigma(\mathcal{C}) = \hat{\mathcal{F}}_\infty$ measurable random variables, which means exactly $\mathcal{M}_0^\infty(\hat{\mathbb{P}}, \hat{\mathbb{F}}) \subset \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$. ∎

Lemma 2.3 *Let W be a d-dimensional $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ local martingale with $W_0 = 0$. The following statements are equivalent :*

1. *$\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W) = \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$, i.e., $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ holds.*
2. *for any $L \in \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$, if $LW_i \in \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ for any $1 \leq i \leq d$, then L is null.*
3. *for any $L \in \mathcal{M}_0^\infty(\hat{\mathbb{P}}, \hat{\mathbb{F}})$, if $LW_i \in \mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ for any $1 \leq i \leq d$, then L is null.*

Proof. This is the consequence of [14, Corollaire(4.12) and Proposition(4.67)] ∎

We recall that the property $\mathfrak{Mrp}$ is invariant by change of probabilities (However, the driving process may change). Suppose $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$. For a probability measure $\hat{\mathbb{Q}}$ locally equivalent to $\hat{\mathbb{P}}$, set

$$\eta_t = \left.\frac{d\hat{\mathbb{Q}}}{d\hat{\mathbb{P}}}\right|_{\hat{\mathcal{F}}_t} , \quad 0 \leq t < \infty.$$

The process η is a strictly positive $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$-martingale. We have the following result.

Lemma 2.4 *Suppose that $\langle W_i, \eta \rangle, 1 \leq i \leq d$, exist in $\hat{\mathbb{F}}$ under $\hat{\mathbb{P}}$. Set*

$$W_{i,t}^{[\eta]} = W_{i,t} - \int_0^t \frac{1}{\eta_{s-}} d\langle \eta, W_i \rangle_s, 0 \leq t < \infty, 1 \leq i \leq d. \tag{6}$$

Then, we have $\mathfrak{Mrp}(\hat{\mathbb{Q}}, \hat{\mathbb{F}}, W^{[\eta]})$.

Proof. Note that $W^{[\eta]}$ is a $(\hat{\mathbb{Q}}, \hat{\mathbb{F}})$ local martingale by Girsanov theorem. By Lemma 2.3, it is enough to prove that, for $L \in \mathcal{M}_{loc,0}(\hat{\mathbb{Q}}, \hat{\mathbb{F}})$, if $LW_i^{[\eta]} \in \mathcal{M}_{loc,0}(\hat{\mathbb{Q}}, \hat{\mathbb{F}})$ for $1 \leq i \leq d$, the process L is null. Set $Y = L\eta$. By Girsanov theorem, Y and $YW_i^{[\eta]}$ belong to $\mathcal{M}_{loc,0}(\hat{\mathbb{P}}, \hat{\mathbb{F}})$. This yields that $\langle Y, W_i \rangle$ exists and $\langle Y - \frac{Y_-}{\eta_-}\cdot\eta, W_i \rangle = 0, 1 \leq i \leq d$. Applying Lemma 2.3 with the $\mathfrak{Mrp}(\hat{\mathbb{P}}, \hat{\mathbb{F}}, W)$ property, we conclude $Y = 0$, and consequently $L = 0$. ∎

3 $\mathfrak{Mrp}$ property before τ

In this section, we assume the setting of the subsection 1.1. We use the notations $\widetilde{X} = X - \Gamma(X)$, $H = \mathbb{1}_{[\tau,\infty)}$ and $L = \mathbb{1}_{\{\tau>0\}}\mathbb{1}_{[\tau,\infty)} - \mathbb{1}_{(0,\tau]}\frac{1}{Z_-}\cdot A$. We need the following lemma from [30]

Lemma 3.1 *We define $Z_{0-} = 1$. Then, $Z_{\tau-} > 0$ on $\{\tau < \infty\}$.*

We introduce

Assumption 3.1   *(i) Assume $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{F}, W)$ for a càdlàg d-dimensional $\mathbb{F}$-adapted process W, and the ellipticity assumption 2.2 for W under $\mathbb{Q}$.*
(ii) Assume $(\mathcal{H}')$ hypothesis.

Here is the main result in this section, which generalizes [21, Lemme(5.15)], [5, Theorem 1] and [4, Theorem 3.3.2] :

Theorem 3.1 *Suppose Assumption 3.1. Then, for any bounded $\zeta \in \mathcal{G}_\tau$, there exist an $\mathbb{F}$-predictable process J and a bounded $\mathbb{F}$-optional process X such that $J\mathbb{1}_{[0,\tau]} \in \mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$ and*

$$\mathbb{Q}[\zeta|\mathcal{G}_t] = \mathbb{Q}[\zeta|\mathcal{G}_0] + \int_0^t (1 - H_{s-})J_s d\widetilde{W}_s \ + \mathbb{1}_{\{\tau>0\}}(\zeta - X_\tau)H_t - \int_0^t K_s(1 - H_{s-})\frac{1}{Z_{s-}}dA_s, \quad (7)$$

for $t \geq 0$, where K is a bounded $\mathbb{F}$-predictable process such that $K_\tau \mathbb{1}_{\{0<\tau<\infty\}} = \mathbb{Q}[(\zeta-X_\tau)\mathbb{1}_{\{0<\tau<\infty\}}|\mathcal{F}_{\tau-}]$.

Proof. The proof can be build from [21, Lemma (5.15)]. However, we prefer to give another proof based on a direct computation of the martingale $\mathbb{Q}[\zeta|\mathcal{G}_t], t \geq 0$.

Consider a bounded random variable $\zeta \in \mathcal{G}_\tau$. From [4], we have the before-default decomposition formula:

$$\mathbb{Q}[\zeta|\mathcal{G}_t] = \frac{\mathbb{Q}[\zeta \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t]}{Z_t}(1 - H_t) + \zeta H_t, \ 0 \leq t < \infty$$

Let $R = \inf\{t \geq 0 : Z_{t-} = 0 \text{ or } Z_t = 0\}$ and, for $n \geq 1$, $R_n = \inf\{t \geq 0 : Z_{t-} < \frac{1}{n} \text{ or } Z_t < \frac{1}{n}\}$. We can check that $\tau \leq R$. Since $Z_{\tau-} > 0$, the process :

$$X_t = \frac{\mathbb{Q}[\zeta \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t]}{Z_t}\mathbb{1}_{\{t<R\}}, \ 0 \leq t < \infty$$

is well defined and bounded. By [1, Lemma 5.2], for any $n \geq 1$, X^{R_n} is a $(\mathbb{Q}, \mathbb{F})$ semimartingale. Hence, the process X is a bounded $(\mathbb{Q}, \mathbb{F})$ semimartingale on the $\mathbb{F}$-predictable set $\mathsf{B} = \cup_{n\geq 1}[0, R_n]$ (see [13, Definition 8.19]). Note that $\sup_{n\geq 1} R_n = R$ which implies $\cup_{n\geq 1}[0, R_n] \supset [0, R)$. Denote the $(\mathbb{Q}, \mathbb{F})$ canonical decomposition on B of X by $X^{[m]} + X^{[v]}$, where $X^{[m]}$ is a $(\mathbb{Q}, \mathbb{F})$ local martingale on B and $X^{[v]}$ is a $\mathbb{F}$-predictable process with finite variation on B. By the property $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{F}, W)$ and ellipticity assumption, there exists an $\mathbb{F}$-predictable process J defined on B such that, for any $n \geq 1$, $J\mathbb{1}_{[0,R_n]} \in \mathcal{I}(\mathbb{Q}, \mathbb{F}, W)$ and

$$X_t^{[m]} = X_0^{[m]} + \int_0^t J_s dW_s, \ 0 \leq t < R$$

Since $J\mathbb{1}_{[0,R_n]}.W_i, 1 \leq i \leq d$, and W are $(\mathbb{Q}, \mathbb{G})$ semimartingales by $(\mathcal{H}')$-Hypothesis, the process $J\mathbb{1}_{[0,R_n]}$ belongs equally to $\mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$ as well as to $\mathcal{I}(\mathbb{Q}, \mathbb{G}, \Gamma(W))$, for every $n \geq 1$ (cf. [21, Proposition(2.1)]). Lemma 2.1 is applicable to $J\mathbb{1}_{[0,R_n]}$ between $\mathbb{F}$ and $\mathbb{G}$.

Let K be a bounded $\mathbb{F}$-predictable process such that

$$K_\tau \mathbb{1}_{\{0<\tau<\infty\}} = \mathbb{Q}[(\zeta - X_\tau)\mathbb{1}_{\{0<\tau<\infty\}}|\mathcal{G}_{\tau-}] = \mathbb{Q}[(\zeta - X_\tau)\mathbb{1}_{\{0<\tau<\infty\}}|\mathcal{F}_{\tau-}]$$

We compute now the martingale $\mathbb{Q}[\zeta|\mathcal{G}_t]$: for $n \geq 1$, for $0 \leq t \leq R_n$

$$
\begin{aligned}
\mathbb{Q}[\zeta|\mathcal{G}_t] &= X_t(1 - H_t) + \zeta H_t \\
&= \int_0^t (1 - H_{s-})dX_s - \int_0^t X_{s-}dH_s + [X, 1 - H]_t + \zeta H_t \\
&= X_0(1 - H_0) + \int_0^t (1 - H_{s-})J_s dW_s + \int_0^t (1 - H_{s-})dX_s^{[v]} + (\zeta - \mathbb{1}_{\{\tau>0\}}X_\tau)H_t \\
&= \mathbb{Q}[\zeta|\mathcal{G}_0] + \int_0^t (1 - H_{s-})J_s d\widetilde{W}_s + \int_0^t (1 - H_{s-})J_s d\Gamma(W)_s + \int_0^t (1 - H_{s-})dX_s^{[v]} \\
&\quad + \mathbb{1}_{\{\tau>0\}}(\zeta - X_\tau)H_t - \int_0^t K_s \mathbb{1}_{\{s\leq\tau\}}\frac{1}{Z_{s-}}dA_s + \int_0^t K_s \mathbb{1}_{\{s\leq\tau\}}\frac{1}{Z_{s-}}dA_s
\end{aligned}
$$

We note that, since $\int_0^t K_s \mathbb{1}_{\{s\leq\tau\}}\frac{1}{Z_{s-}}dA_s$ is the $(\mathbb{Q}, \mathbb{G})$ predictable dual projection of $\mathbb{1}_{\{\tau>0\}}(\zeta - X_\tau)H$ (see [21]), the process

$$\mathbb{1}_{\{\tau>0\}}(\zeta - X_\tau)H_t - \int_0^t K_s \mathbb{1}_{\{s\leq\tau\}}\frac{1}{Z_{s-}}dA_s$$

is a $(\mathbb{Q}, \mathbb{G})$ local martingale. This implies that the following $\mathbb{G}$-predictable process with finite variation

$$\int_0^t (1 - H_{s-})J_s d\Gamma(W)_s + \int_0^t (1 - H_{s-})dX_s^{[v]} + \int_0^t K_s \mathbb{1}_{\{s \leq \tau\}} \frac{1}{Z_{s-}} dA_s$$

is a $(\mathbb{Q}, \mathbb{G})$ local martingale, so that it is null. Consequently,

$$\mathbb{Q}[\zeta|\mathcal{G}_t] = \mathbb{Q}[\zeta|\mathcal{G}_0] + \int_0^t (1 - H_{s-})J_s d\widetilde{W}_s + \mathbb{1}_{\{\tau > 0\}}(\zeta - X_\tau)H_t - \int_0^t K_s \mathbb{1}_{\{s \leq \tau\}} \frac{1}{Z_{s-}} dA_s \tag{8}$$

for $0 \leq t \leq R_n$, $n \geq 1$.

Let us prove that $\mathbb{Q}[\forall n \geq 1, R_n < \tau] = 0$. Actually, on the set $\{\forall n \geq 1, R_n < \tau\}$, there can exist two situations. Firstly the sequence (R_n) is stationary, i.e., for some n, $R = R_n < \tau$. It is impossible because $Z_{R+\epsilon} = 0, \forall \epsilon > 0$ (see [13, Theorem 2.62]) whilst $Z_{\tau-} > 0$. When the sequence (R_n) is not stationary, we must have $R = \lim_n R_n = \tau$ and $Z_{R-} = 0$. Once again it is impossible because $Z_{\tau-} > 0$.

We conclude that $[0, \tau] \subset$ B. Hence, the process J is well defined on $[0, \tau]$ and the above formula (8) is true on $[0, \tau]$. Note that $\mathbb{Q}[\zeta|\mathcal{G}_t]$ and $\mathbb{1}_{\{\tau > 0\}}(\zeta - X_\tau)H_t - \int_0^t K_s \mathbb{1}_{\{s \leq \tau\}} \frac{1}{Z_{s-}} dA_s$ are $(\mathbb{Q}, \mathbb{G})$ local martingales on the whole $\mathbb{R}_+$. Consequently, $J\mathbb{1}_{[0,R_n]}$ is in $\mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$ uniformly for all $n \geq 1$, which implies $J\mathbb{1}_{[0,\tau]} \in \mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$. The theorem is proved. ∎

Remark 3.1 The formula (7) can also be written as

$$\begin{aligned}
\mathbb{Q}[\zeta|\mathcal{G}_t] &= \mathbb{Q}[\zeta|\mathcal{G}_0] + \int_0^t \mathbb{1}_{\{0 < s \leq \tau\}} J_s d\widetilde{W}_s \\
&\quad + \int_0^t K_s \mathbb{1}_{\{0 < s \leq \tau\}} dL_s + \mathbb{1}_{\{\tau > 0\}}(\zeta - X_\tau - K_\tau)H_t
\end{aligned} \tag{9}$$

This shows that, for any bounded $(\mathbb{Q}, \mathbb{G})$-martingale Y, the stopped martingale Y^τ is the sum of a stochastic integral against $(\widetilde{W}, L)$ and a $(\mathbb{Q}, \mathbb{G})$-martingale of the form ξH, where $\xi \in \mathcal{G}_\tau$ such that $\mathbb{Q}[\xi|\mathcal{G}_{\tau-}] = 0$. This remark links the formula (7) to [21, Théorème(5.12)]

Suppose now that $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau = \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$ under $\mathbb{Q}$. In this case, $\mathbb{1}_{\{0 < \tau < \infty\}} K_\tau = \mathbb{1}_{\{0 < \tau < \infty\}}(\zeta - X_\tau)$ and formula (7) writes as

$$\mathbb{Q}[\zeta|\mathcal{G}_t] = \mathbb{Q}[\zeta|\mathcal{G}_0] + \int_0^t \mathbb{1}_{\{0 < s \leq \tau\}} J_s d\widetilde{W}_s + \int_0^t K_s \mathbb{1}_{\{0 < s \leq \tau\}} dL_s \tag{10}$$

From this identity, appplying Lemma 2.2, we conclude that the property $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^\tau, (\widetilde{W}^\tau, L))$ holds, where $\mathbb{G}^\tau$ denotes the stopped filtration $(\mathcal{G}_{t \wedge \tau} : t \geq 0)$, and $\widetilde{W}^\tau$ is $\widetilde{W}$ stopped at τ. We can state this conclusion in another way :

Theorem 3.2 *Suppose Assumption 3.1. Then, $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^\tau, (\widetilde{W}^\tau, L))$ and $\mathbb{1}_{\{0 < \tau < \infty\}} W_\tau \in \mathcal{G}_{\tau-}$ hold, if and only if $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau = \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$ under $\mathbb{Q}$.*

Proof. We have proved that the condition is sufficient. Suppose now $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^\tau, (\widetilde{W}^\tau, L))$ and $\mathbb{1}_{\{0 < \tau < \infty\}} W_\tau \in \mathcal{G}_{\tau-}$. Any $(\mathbb{Q}, \mathbb{G}^\tau)$ bounded martingale Y with $Y_0 = 0$ has a representation as in (10). We have, therefore,

$$\begin{aligned}
&\mathbb{1}_{\{0 < \tau < \infty\}} \Delta_\tau Y \\
&= J_\tau \mathbb{1}_{\{0 < \tau < \infty\}} \Delta_\tau \widetilde{W} + \mathbb{1}_{\{0 < \tau < \infty\}} K_\tau (1 - \tfrac{1}{Z_{\tau-}} \Delta_\tau A) \\
&= J_\tau \mathbb{1}_{\{0 < \tau < \infty\}}(W_\tau - W_{\tau-}) - J_\tau \mathbb{1}_{\{0 < \tau < \infty\}} \Delta_\tau \Gamma(W) + \mathbb{1}_{\{0 < \tau < \infty\}} K_\tau (1 - \tfrac{1}{Z_{\tau-}} \Delta_\tau A)
\end{aligned}$$

where $J, K, Z_-, \Delta\Gamma(W), \Delta A$ are $\mathbb{F}$-predictable processes and $\{0 < \tau < \infty\} \in \mathcal{G}_{\tau-}$. Consequently, $\mathbb{1}_{\{0 < \tau < \infty\}} Y_\tau = \mathbb{1}_{\{0 < \tau < \infty\}} Y_{\tau-} + \mathbb{1}_{\{0 < \tau < \infty\}} \Delta_\tau Y \in \mathcal{G}_{\tau-}$, and (cf. [8, Chapter XX section 22])

$$\begin{aligned}
&\{0 < \tau < \infty\} \cap \mathcal{G}_\tau \\
&= \{0 < \tau < \infty\} \cap \sigma\{\tau, \mathbb{1}_{\{0 < \tau < \infty\}} Y_\tau : Y \text{ is a } (\mathbb{Q}, \mathbb{G}^\tau) \text{ bounded martingale}\} \\
&\subset \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}
\end{aligned}$$

The inverse inclusion $\mathcal{G}_{\tau-} \subset \mathcal{G}_\tau$ is obvious. The theorem is proved. ∎

Remark 3.2 About the condition $\mathbb{1}_{\{0<\tau<\infty\}}W_\tau \in \mathcal{G}_{\tau-}$, it is satisfied, if W is continuous. It is also satisfied, if the random time τ avoids the $\mathbb{F}$ stopping times, since then $W_\tau = W_{\tau-} \in \mathcal{G}_{\tau-}$.

4 $\mathfrak{M}$rp property after the default time τ

We work in the setting of the subsection 1.1.

4.1   Fragments of the filtration $\mathbb{G}$

Definition 4.1 *For a $\mathbb{G}$-stopping time T, we define the σ-algebra*

$$\mathcal{G}_T^* = \{T < \tau\} \cap \mathcal{F}_T + \{\tau \leq T < \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_T) + \{T = \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_\infty)$$

For two $\mathbb{G}$-stopping times S, T such that $S \leq T$, we define the family $\mathbb{G}^{(S,T]}$ of σ-algebras :

$$\mathcal{G}_t^{(S,T]} = \{\{T \leq S \vee t\} \cap A + \{S \vee t < T\} \cap B : A \in \mathcal{G}_T^*, \ B \in \mathcal{G}_{S \vee t}\}, \ 0 \leq t < \infty.$$

For general $\mathbb{G}$-stopping times S, T, we define the family $\mathbb{G}^{(S,T]} = \mathbb{G}^{(S,S \vee T]}$.

For a process X, for $\mathbb{G}$-stopping times S, T such that $S \leq T$, we denote $X_t^{(S,T]} = X_{(S \vee t) \wedge T} - X_S, 0 \leq t < \infty$. For general $\mathbb{G}$-stopping times S, T, we define $X^{(S,T]} = X^{(S,S \vee T]}$.

The following results exhibit the properties of the family $\mathbb{G}^{(S,T]}$ in relation with the filtration $\mathbb{G}$. The proofs will be given in the Appendix.

Lemma 4.1 *For any $\mathbb{G}$-stopping times S, T, we have $\mathcal{G}_{T-} \subset \mathcal{G}_T^* \subset \mathcal{G}_T$ and*

$$\mathcal{G}_{S \vee T}^* = \{S < T\} \cap \mathcal{G}_T^* + \{T \leq S\} \cap \mathcal{G}_S^*$$

Proposition 4.1 *We consider two $\mathbb{G}$-stopping times S, T such that $S \leq T$. We have*
 (1) $\mathbb{G}^{(S,T]}$ is a right-continuous filtration
 (2) For a $\mathbb{G}^{(S,T]}$-stopping time R such that $S \leq R \leq T$, we have

$$\mathcal{G}_R^{(S,T]} \quad = \quad \{R = T\} \cap \mathcal{G}_T^* + \{R < T\} \cap \mathcal{G}_R$$

 In particular, $\mathcal{G}_S^{(S,T]} = \mathcal{G}_0^{(S,T]}$ and $\mathcal{G}_T^{(S,T]} = \mathcal{G}_T^*$.
 (3) For any $\mathbb{G}$-adapted process X such that $X_T \in \mathcal{G}_T^*$, $X^{(S,T]}$ is a $\mathbb{G}^{(S,T]}$-adapted process. Conversely, for any $\mathbb{G}^{(S,T]}$-adapted process X', $X'^{(S,T]}$ is a $\mathbb{G}$-adapted process.
 (4) For any $\mathbb{G}$-predictable process K, $K^{(S,T]}$ and $\mathbb{1}_{(S,T]}K$ define $\mathbb{G}^{(S,T]}$-predictable processes. Conversely, for any $\mathbb{G}^{(S,T]}$-predictable process K', $K'^{(S,T]}$ and $\mathbb{1}_{(S,T]}K'$ define $\mathbb{G}$-predictable processes.
 (5) For any $(\mathbb{Q}, \mathbb{G})$ local martingale X such that $X_T \in \mathcal{G}_T^*$, $X^{(S,T]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ local martingale. Conversely, for any $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ local martingale X', $X'^{(S,T]}$ is a $(\mathbb{Q}, \mathbb{G})$ local martingale.
 (6) For any $\mathbb{G}$-predictable process K, for any $(\mathbb{Q}, \mathbb{G})$ local martingale X such that $X_T \in \mathcal{G}_T^*$, the fact that $\mathbb{1}_{(S,T]}K$ is X-integrable in $\mathbb{G}$ implies that $\mathbb{1}_{(S,T]}K$ is $X^{(S,T]}$-integrable in $\mathbb{G}^{(S,T]}$. Conversely, for any $\mathbb{G}^{(S,T]}$-predictable process K', for any $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ local martingale X', the fact that $\mathbb{1}_{(S,T]}K'$ is X'-integrable in $\mathbb{G}^{(S,T]}$ implies that $\mathbb{1}_{(S,T]}K'$ is $X'^{(S,T]}$-integrable in $\mathbb{G}$.

4.2 $\mathscr{sH}$-measures

Definition 4.2 *Let S, T be $\mathbb{G}$-stopping times. A probability measure $\mathbb{Q}'$ defined on $\mathcal{G}_\infty$ is called a $\mathscr{sH}$-measure over the random time interval $(S, T]$ (with respect to $(\mathbb{Q}, \mathbb{G})$), if $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ on $\mathcal{G}_\infty$, and if, for any $(\mathbb{Q}, \mathbb{F})$ local martingale X, $X^{(S,T]}$ is a $(\mathbb{Q}', \mathbb{G}^{(S,T]})$ local martingale.*

Remark 4.1 Note that, since obviously $X_T \in \mathcal{G}_T^*$, $X^{(S,T]}$ is $\mathbb{G}^{(S,T]}$-adapted (Proposition 4.1 (3)). The notion of $\mathscr{sH}$-measure resembles to the immersion condition, but not exactly, because $\mathbb{Q} \neq \mathbb{Q}'$ and $\mathbb{F} \not\subseteq \mathbb{G}^{(S,T]}$. The notation $\mathscr{sH}$ refers to a "skewed" immersion condition.

Remark 4.2 Recall that $\mathscr{sH}$ measure condition can be used to establish the $(\mathcal{H}')$ hypothesis. See [29].

Theorem 4.1 *Suppose Assumption 3.1. Then, for any $\mathbb{F}$-stopping time T, for any $\mathbb{G}$-stopping time S such that $S \geq \tau$, for any $\mathscr{sH}$-measure $\mathbb{Q}'$ over $(S, T]$, $\mathfrak{Mrp}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$ holds.*

Proof. We suppose that the set $\{T > S\}$ is not empty, because otherwise, nothing is to be proved. The proof is presented in several steps.

▶ Let ζ be an $\mathcal{F}_T$-measurable bounded random variable, and X be the martingale $X_t = \mathbb{Q}[\zeta | \mathcal{F}_t], 0 \leq t < \infty$. T being a $\mathbb{F}$-stopping time, we have the identity $\zeta = X_{T \vee t}, t \geq 0$, a key point in the computations below. By the $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{F}, W)$ property with the ellipticity assumption, X has the following representation in the filtration $\mathbb{F}$ under $\mathbb{Q}$ (see (5)) :

$$X_t = X_0 + \int_0^t J_s dW_s = X_0 + \int_0^{t \wedge T} J_s dW_s, \ 0 \leq t < \infty \tag{11}$$

where J is a process in $\mathcal{I}(\mathbb{Q}, \mathbb{F}, W)$.

▶ Since W and $J_i \cdot W_i, 1 \leq i \leq d$, are $(\mathbb{Q}, \mathbb{G})$ semimartingales, according to [21, Proposition(2.1)], for any $1 \leq i \leq d$, the stochastic integral $\int_0^t |J_{i,s}||d\Gamma(W)_s|, t \geq 0$, defines a finite valued process, and J is integrable with respect to W in the sense of $(\mathbb{Q}, \mathbb{G})$ semimartingale. This together with Lemma 2.1 entails that the formula (11) is also valid in the filtration $\mathbb{G}$.

▶ We are in particular interested in a variant of the formula (11) :

$$X_t^{(S,T]} = \int_0^t \mathbb{1}_{\{S < s \leq T\}} J_s dW_s^{(S,T]}, \ 0 \leq t < \infty \tag{12}$$

Then, we check straightforwardly that $J\mathbb{1}_{(S,T]}$ is integrable with respect to $W^{(S,T]}$ in the sense of $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ semimartingale and the formula (12) is also valid in the filtration $\mathbb{G}^{(S,T]}$ under $\mathbb{Q}$. (cf. Proposition 4.1 (6) and Lemma 2.1.)

▶ Let $\mathbb{Q}'$ be a $\mathscr{sH}$-measure on $(S, T]$. The two probabilities $\mathbb{Q}'$ and $\mathbb{Q}$ are equivalent on $\mathcal{G}_\infty$. The processes $W^{(S,T]}$ and $X^{(S,T]}$ are $(\mathbb{Q}', \mathbb{G}^{(S,T]})$ local martingales. By the ellipticity assumption, for $1 \leq i \leq d$, for $t \geq 0$, we have

$$\int_0^t \mathbb{1}_{\{S < s \leq T\}} J_{i,s}^2 d[W_i^{(S,T]}]_s \leq C \sum_{k,j=1}^d \int_0^t \mathbb{1}_{\{S < s \leq T\}} J_{k,s} J_{j,s} d[W_k, W_j]_s = C[X^{(S,T]}]_t,$$

$\mathbb{Q} - a.s.$ as well as $\mathbb{Q}' - a.s..$ We note that the above bracket is the same under $\mathbb{Q}$ or under $\mathbb{Q}'$, in $\mathbb{G}$ or in $\mathbb{G}^{(S,T]}$. Note that $[X^{(S,T]}]$ is $\mathbb{G}^{(S,T]}$-adapted (cf. Proposition 4.1 (3)). As X is bounded, the bracket $[X^{(S,T]}]$ is $\mathbb{G}^{(S,T]}$-locally bounded, which entails that the processes $\mathbb{1}_{(S,T]} J_i^2 \cdot [W_i^{(S,T]}]$ are $\mathbb{G}^{(S,T]}$-locally

bounded. This means $\mathbb{1}_{(S,T]}J \in \mathcal{I}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$. From Lemma 2.1, we conclude that the formula (12) is also valid in the filtration $\mathbb{G}^{(S,T]}$ under $\mathbb{Q}'$.

▶ Consider now a bounded Borel function g on $[0, \infty]$. Since $\tau \leq S$, we have $g(\tau) \in \mathcal{G}_S^* \subset \mathcal{G}_0^{(S,T]}$. Therefore, the process $g(\tau)\mathbb{1}_{(S,T]}J$ is a $\mathbb{G}^{(S,T]}$-predictable process whilst $g(\tau)X^{(S,T]}$ is a $(\mathbb{Q}', \mathbb{G}^{(S,T]})$ local martingale. Since g is bounded, $g(\tau)\mathbb{1}_{(S,T]}J \in \mathcal{I}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$. We can apply the formula (12) and write:

$$g(\tau)X_t^{(S,T]} = \int_0^t g(\tau)\mathbb{1}_{\{S < s \leq T\}}J_s dW_s^{(S,T]}, \ t \geq 0.$$

valid in $\mathbb{G}^{(S,T]}$ under $\mathbb{Q}'$.

▶ As $g(\tau)X^{(S,T]}$ is bounded, $g(\tau)X^{(S,T]}$ and $g(\tau)J\mathbb{1}_{(S,T]} \cdot W^{(S,T]}$ are bounded true $(\mathbb{Q}', \mathbb{G}^{(S,T]})$ martingales (in particular, they are $\mathcal{H}_0^1(\mathbb{Q}', \mathbb{G}^{(S,T]})$ martingales). Noting that $X_t = X_T = \zeta$ for $t \geq T$ and $g(\tau)X_S \in \mathcal{G}_0^{(S,T]}$, we obtain

$$
\begin{aligned}
& g(\tau)X_S + \int_0^t g(\tau)\mathbb{1}_{\{S < s \leq T\}}J_s dW_s^{(S,T]} \\
= \ & \mathbb{Q}'[g(\tau)X_S + \int_0^\infty g(\tau)\mathbb{1}_{\{S < s \leq T\}}J_s dW_s^{(S,T]} \ |\mathcal{G}_t^{(S,T]}] \\
= \ & \mathbb{Q}'[g(\tau)\zeta \ |\mathcal{G}_t^{(S,T]}], t \geq 0,
\end{aligned}
$$

We deduce from this identity that $\mathbb{Q}'[g(\tau)\zeta \ |\mathcal{G}_0^{(S,T]}] = g(\tau)X_S$ and hence

$$\dagger(g(\tau)\zeta)_t = \int_0^t g(\tau)\mathbb{1}_{\{S < s \leq T\}}J_s dW_s^{(S,T]}, \ t \geq 0$$

valid in $\mathbb{G}^{(S,T]}$ under $\mathbb{Q}'$, where we use the operator $\dagger$ (as it is defined in Lemma 2.2) with respect to $(\mathbb{Q}', \mathbb{G}^{(S,T]})$. This identity shows that $\dagger(g(\tau)\zeta)$ belongs to $\mathcal{M}_{loc,0}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$.

▶ Let $\mathcal{C}$ denote the class of all set of the form

$$\{S < T\} \cap \{s < \tau \leq t\} \cap A + \{T \leq S\} \cap B$$

where $0 \leq s < t$ and $A \in \mathcal{F}_T$ and $B \in \mathcal{G}_S^*$. Note that $\{S < T\}, \{T \leq S\} \cap B \in \mathcal{G}_S^{(S,T]} = \mathcal{G}_0^{(S,T]}$ according to Proposition 4.1 (2). This observation together with what we have proved previously shows that, for any $F \in \mathcal{C}$, the $(\mathbb{Q}', \mathbb{G}^{(S,T]})$ martingale $\dagger F$ belongs to $\mathcal{M}_{loc,0}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$. On the other hand, the class $\mathcal{C}$ is a π-system and $\sigma(\mathcal{C}) = \mathcal{G}_\infty^{(S,T]}$ because of Lemma 4.1 and $\tau \leq S$. We apply Lemma 2.2, and we conclude the property $\mathfrak{Mrp}(\mathbb{Q}', \mathbb{G}^{(S,T]}, W^{(S,T]})$ ∎

4.3 Local solution and global solution

Theorem 4.2 *Suppose Assumption 3.1. Suppose that*

*$\mathscr{H}$ **measure condition covering** (τ, ∞) : There exists a countable family of $\mathbb{G}$-stopping times $\{S_j, T_j : j \in \mathbb{N}\}$, such that*

1. *for any $j \in \mathbb{N}$, there exists a $\mathscr{H}$-measure $\mathbb{Q}_j$ over the time interval $(S_j, T_j]$.*
2. *T_j are $\mathbb{F}$-stopping times.*
3. *$S_j \geq \tau$ and $(\tau, \infty) = \cup_{i \in \mathbb{N}}(S_j, T_j)$.*

Then, $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(\tau, \infty]}, \widetilde{W}^{(\tau, \infty]})$ holds.

Remark 4.3 Note that $\mathcal{G}_t^{(\tau, \infty]} = \mathcal{G}_{\tau \vee t}, t \geq 0$.

Proof. The proof consists to prove firstly $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(S_j, T_j]}, \widetilde{W}^{(S_j, T_j]})$ (our local solutions), and then to pass from local solution to the global one. It is divided into several steps.

1) Local solutions

▶ Let $j \in \mathbb{N}$ be fixed and $\eta_t = \frac{d\mathbb{Q}}{d\mathbb{Q}_j}\big|_{\mathcal{G}_t^{(S_j,T_j]}}, t \geq 0$. (We recall that $\mathcal{G}_\infty^{(S_j,T_j]} = \mathcal{G}_{S_j \vee T_j}^*$) In this step, we prove that $\langle \eta, W_i^{(S_j,T_j]}\rangle$, $1 \leq i \leq d$, exists in $\mathbb{G}^{(S_j,T_j]}$ under $\mathbb{Q}_j$. Actually, $W^{(S_j,T_j]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ semimartingale whose drift is the $\mathbb{G}^{(S_j,T_j]}$-predictable process with finite variation $\Gamma(W)^{(S_j,T_j]}$. Hence, $W^{(S_j,T_j]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ special semimartingale. There exists an increasing sequence of $\mathbb{G}^{(S_j,T_j]}$-stopping times (R'_n) converging $\mathbb{Q}$-almost surely to infinity, such that $W_{R'_n}^*$ is $\mathbb{Q}$-integrable for any $n \geq 1$, where $W_t^* = \sup_{s \leq t} |W_s^{(S_j,T_j]}|, t \geq 0$ ($|\cdot|$ denotes the Euclidean space norm). Let

$$
R_n = \begin{cases} R'_n & R'_n < S_j \vee T_j \\ \\ \infty & R'_n \geq S_j \vee T_j \end{cases}
$$

Since $W_t^{(S_j,T_j]} = W_{S_j \vee T_j}^{(S_j,T_j]}$ for $t \geq S_j \vee T_j$, $W_{R_n}^*$ is also $\mathbb{Q}$-integrable. This is equivalent to say that, for any $\mathbb{G}^{(S_j,T_j]}$-stopping time R, $\eta_{R \wedge R_n} W_{R \wedge R_n}^*$ is $\mathbb{Q}_j$-integrable. We now fix a $1 \leq i \leq d$ and set, for $m \in \mathbb{N}$,

$$
U_m = \inf\{s : |(\eta W_i^{(S_j,T_j]})_s| > m\}
$$

We have

$$
|(\eta W_i^{(S_j,T_j]})_{U_m \wedge R_n}^*| \leq m + \eta_{U_m \wedge R_n} W_{U_m \wedge R_n}^*
$$

where the right hand term is $\mathbb{Q}_j$-integrable for any $n \geq 1$. It is clear that U_m tends $\mathbb{Q}_j$-almost surely to infinity. On the other hand, for any $0 < K < \infty$, since $\{R'_n < K, R'_n < S_j \vee T_j\} \in \mathcal{G}_{S_j \vee T_j-}^{(S_j,T_j]} \subset \mathcal{G}_{S_j \vee T_j}^*$ (cf [13, Theorem 3.4]) for any $n \geq 1$, we have

$$
\lim_{n \to \infty} \mathbb{Q}_j[\eta_{S_j \vee T_j} \mathbb{1}_{\{R_n < K\}}] = \lim_{n \to \infty} \mathbb{Q}_j[\eta_{S_j \vee T_j} \mathbb{1}_{\{R'_n < K, R'_n < S_j \vee T_j\}}] = \lim_{n \to \infty} \mathbb{Q}[R'_n < K, R'_n < S_j \vee T_j] = 0
$$

which means that R_n tends to infinity $\mathbb{Q}_j$-almost surely. Now the integration by parts formula gives

$$
[\eta, W_i^{(S_j,T_j]}]_t = (\eta W_i^{(S_j,T_j]})_t - (\eta W_i^{(S_j,T_j]})_0 - \eta_- \bullet (W_i^{(S_j,T_j]})_t - (W_i^{(S_j,T_j]})_- \bullet \eta_t, t \geq 0.
$$

$\eta_- \bullet (W^{(S_j,T_j]})$ and $(W^{(S_j,T_j]})_- \bullet \eta$ are $(\mathbb{Q}_j, \mathbb{G}^{(S_j,T_j]})$ local martingales, and hence locally integrable. Reducing U_m if necessary, we can assume that the sequence $(U_m)_{m \geq 1}$ makes these two local martingales locally integrable in $(\mathbb{Q}_j, \mathbb{G}^{(S_j,T_j]})$. In this way we prove that $[\eta, W^{(S_j,T_j]}]$ is $\mathbb{G}^{(S_j,T_j]}$-locally $\mathbb{Q}_j$-integrable, and hence $\langle \eta, W_i^{(S_j,T_j]}\rangle$ exists in $(\mathbb{Q}_j, \mathbb{G}^{(S_j,T_j]})$.

▶ According to Theorem 4.1, $\mathfrak{Mrp}(\mathbb{Q}_j, \mathbb{G}^{(S_j,T_j]}, W^{(S_j,T_j]})$ holds. By Lemma 2.4, $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]}, W^{[\eta],(S_j,T_j]})$ holds, where $W_i^{[\eta],(S_j,T_j]} = W_i^{(S_j,T_j]} - \frac{1}{\eta_-} \bullet \langle \eta, W_i^{(S_j,T_j]}\rangle, 1 \leq i \leq d$, is a $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ local martingale. On the other hand, the processes

$$
\widetilde{W}_i^{(S_j,T_j]} = (W_i - \Gamma(W_i))^{(S_j,T_j]} = W_i^{(S_j,T_j]} - \Gamma(W_i)^{(S_j,T_j]}
$$

are also $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ local martingales (cf. Proposition 4.1 (5)). This implies that $\frac{1}{\eta_-} \bullet \langle \eta, W_i^{(S_j,T_j]}\rangle - \Gamma(W_i)^{(S_j,T_j]}$ are $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ predictable local martingales with finite variation. Hence, $\frac{1}{\eta_-} \bullet \langle \eta, W_i^{(S_j,T_j]}\rangle = \Gamma(W_i)^{(S_j,T_j]}$ under $\mathbb{Q}$. In other words, $W_i^{[\eta],(S_j,T_j]} = \widetilde{W}_i^{(S_j,T_j]}$ and $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]}, \widetilde{W}^{(S_j,T_j]})$ holds. We get our local solutions.

2) Global solution

▶ Let us now prove the global solution $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(\tau,\infty]}, \widetilde{W}^{(\tau,\infty]})$. We use Lemma 2.3. Let N be any element in $\mathcal{M}_0^\infty(\mathbb{Q}, \mathbb{G}^{(\tau,\infty]})$ such that $N\widetilde{W}_i^{(\tau,\infty]} \in \mathcal{M}_{loc,0}(\mathbb{Q}, \mathbb{G}^{(\tau,\infty]})$, $1 \leq i \leq d$. We will prove that $N \equiv 0$, which will achieve the proof of the theorem.

A "natural idea" would be to say that $N^{(S_j,T_j]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ local martingale, so that $N^{(S_j,T_j]}$ takes the form : $N^{(S_j,T_j]} = J.\widetilde{W}^{(S_j,T_j]}$ because of $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]}, \widetilde{W}^{(S_j,T_j]})$. This would entail

$$[N^{(S_j,T_j]}] = J\mathbb{1}_{(S_j,T_j]}.[N,\widetilde{W}] \in \mathcal{M}_0^\infty(\mathbb{Q}, \mathbb{G}^{(\tau,\infty]})$$

But $[N^{(S_j,T_j]}]$ is an increasing process. Hence, $[N^{(S_j,T_j]}]$ had to be null. We could then conclude $N \equiv 0$ by the covering condition.

However, this "natural idea" can not work because we can not guarantee that $N^{(S_j,T_j]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$ local martingale. Proposition 4.1 (5) can not be applied here, because we do not know if $N_{S_j \vee T_j} \in \mathcal{G}^*_{S_j \vee T_j}$. The main obstacle is due to the possible jumps of N. That is why our proof concentrates largely on the study of ΔN.

▶ We note that N is a local martingale in the filtration $\mathbb{G}$. In fact, $N_\tau = N_0 = 0$ because $\mathcal{G}_0^{(\tau,\infty]} = \mathcal{G}_\tau^{(\tau,\infty]}$ according to Proposition 4.1 (2). This entails $N = N^{(\tau,\infty]}$. According to Proposition 4.1 (5), N is a $(\mathbb{Q}, \mathbb{G})$ local martingale (as N is bounded, it is in fact a true martingale). For the same reason $N\widetilde{W}_i^{(\tau,\infty]}$ is a $(\mathbb{Q}, \mathbb{G})$ local martingale. Applying the integration by parts formula, we see that $[N, \widetilde{W}_i] \in \mathcal{M}_{loc,0}(\mathbb{Q}, \mathbb{G})$. Taking the stochastic integrals, we obtain that, for any process $J \in \mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$,

$$[N, J.\widetilde{W}] \in \mathcal{M}_{loc,0}(\mathbb{Q}, \mathbb{G}). \tag{13}$$

▶ Let us study the jumps process ΔN at predictable times. Let T be any $\mathbb{G}$-predictable stopping time. We have
$$[\mathbb{1}_{[T]}.N, J.\widetilde{W}] = [N, \mathbb{1}_{[T]}J.\widetilde{W}] \in \mathcal{M}_{loc,0}(\mathbb{Q}, \mathbb{G})$$

Consider the $(\mathbb{Q}, \mathbb{G})$ local martingale $X = \mathbb{1}_{[T]}.N$. We have $X = \mathbb{1}_{\{\tau < T\}}\Delta_T N \mathbb{1}_{[T,\infty)}$ and, for any $j \in \mathbb{N}$, $\Delta_{T_j}X^{T_j} = \mathbb{1}_{\{T=T_j\}}\Delta_T N$.

Let $j \in \mathbb{N}$ be fixed. Set $\varkappa = \mathbb{1}_{\{T=T_j\}}\Delta_T N$ and $\varkappa_- = \mathbb{Q}[\varkappa | \mathcal{G}_{T-}]$. Using the fact that T is $\mathbb{G}$-predictable, by a direct computation, we obtain

$$(\varkappa \mathbb{1}_{[T_j,\infty)})^{(p)} = \mathbb{1}_{\{T \leq T_j\}}\varkappa_- \mathbb{1}_{[T,\infty)}$$

where $(\varkappa \mathbb{1}_{[T_j,\infty)})^{(p)}$ denotes the $(\mathbb{Q}, \mathbb{G})$ predictable dual projection of the jump process $\varkappa \mathbb{1}_{[T_j,\infty)}$.

Set $X' = \varkappa \mathbb{1}_{[T_j,\infty)} - (\varkappa \mathbb{1}_{[T_j,\infty)})^{(p)}$ and

$$X'' = X^{T_j} - X' = (\mathbb{1}_{\{T<T_j\}}\Delta_T N + \mathbb{1}_{\{T \leq T_j\}}\varkappa_-)\mathbb{1}_{[T,\infty)}.$$

Since $X'' = X''^{T_j}$ and T is $\mathbb{G}$-predictable,

$$X''_{S_j \vee T_j} = X''_{T_j} = \mathbb{1}_{\{T<T_j\}}\Delta_T N + \mathbb{1}_{\{T \leq T_j\}}\varkappa_- \in \mathcal{G}_{T_j-} \subset \mathcal{G}_{S_j \vee T_j-} \subset \mathcal{G}^*_{S_j \vee T_j}$$

Proposition 4.1 (5) implies that $X''^{(S_j,T_j]} \in \mathcal{M}_{loc,0}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$. By $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]}, \widetilde{W}^{(S_j,T_j]})$, there exists a d-dimensional process $J \in \mathcal{I}(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]}, \widetilde{W}^{(S_j,T_j]})$ such that $X''^{(S_j,T_j]} = J.\widetilde{W}^{(S_j,T_j]}$ in $(\mathbb{Q}, \mathbb{G}^{(S_j,T_j]})$. Applying Proposition 4.1 (4) and (6) and Lemma 2.1, we have $J\mathbb{1}_{(S_j,T_j]} \in \mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$ and

$$X''^{(S_j,T_j]} = J\mathbb{1}_{(S_j,T_j]}.\widetilde{W}^{(S_j,T_j]} = J\mathbb{1}_{(S_j,T_j]}.\widetilde{W}$$

in $(\mathbb{Q}, \mathbb{G})$. From this relation, in computing the jump at T, we deduce

$$(\mathbb{1}_{\{T<T_j\}}\Delta_T N + \mathbb{1}_{\{T \leq T_j\}}\varkappa_-)\mathbb{1}_{\{S_j < T \leq T_j\}} = J_T \mathbb{1}_{\{S_j < T \leq T_j\}}\Delta_T \widetilde{W}$$

We now compute the bracket between N and $J\mathbb{1}_{(S_j,T_j]}.\widetilde{W}$ (see (13)) :

$$[\mathbb{1}_{[T]}.N, J\mathbb{1}_{(S_j,T_j]}.\widetilde{W}] = (\mathbb{1}_{\{T<T_j\}}(\Delta_T N)^2 + \Delta_T N \mathbb{1}_{\{T \leq T_j\}}\varkappa_-)\mathbb{1}_{\{S_j < T \leq T_j\}}\mathbb{1}_{[T,\infty)}$$

We know that $[\mathbb{1}_{[T]}\boldsymbol{.}N, J\mathbb{1}_{(S_j,T_j]}\boldsymbol{.}\widetilde{W}]$ is a $(\mathbb{Q},\mathbb{G})$ local martingale. Let R be any $\mathbb{G}$-stopping time reducing this local martingale to a uniformly integrable $(\mathbb{Q},\mathbb{G})$ martingale. As the local martingale is null at origin, we have

$$
\begin{aligned}
0 &= \mathbb{Q}[[\mathbb{1}_{[T]}\boldsymbol{.}N, J\mathbb{1}_{(S_j,T_j]}\boldsymbol{.}\widetilde{W}]_R]\\
&= \mathbb{Q}[(\mathbb{1}_{\{T<T_j\}}(\Delta_T N)^2 + \Delta_T N\mathbb{1}_{\{T\leq T_j\}}\varkappa_-)\mathbb{1}_{\{S_j<T\leq T_j\}}\mathbb{1}_{\{T\leq R\}}]\\
&= \mathbb{Q}[(\Delta_T N)^2\mathbb{1}_{\{S_j<T<T_j\}}\mathbb{1}_{\{T\leq R\}}]
\end{aligned}
$$

where $\mathbb{Q}[\Delta_T N\mathbb{1}_{\{T\leq T_j\}}\varkappa_-\mathbb{1}_{\{S_j<T\leq T_j\}}\mathbb{1}_{\{T\leq R\}}] = 0$, because $\{S_j < T \leq T_j\} \in \mathcal{G}_{T-}, \varkappa_- \in \mathcal{G}_{T-}, \{T \leq R\} \in \mathcal{G}_{T-}$, and T is $\mathbb{G}$-predictable, and N is a bounded $(\mathbb{Q},\mathbb{G})$ martingale. This expectation nullity implies the jump nullity $\Delta_T N\mathbb{1}_{\{S_j<T<T_j\}} = 0$.

This nullity being true for any $j \in \mathbb{N}$, we can apply the covering condition and we conclude $\Delta_T N = \Delta_T N\mathbb{1}_{\{\tau<T<\infty\}} = 0$.

▶ We have proved that N has no jumps at $\mathbb{G}$-predictable times. For $j \in \mathbb{N}$, we introduce the process

$$
N' = \Delta_{S_j\vee T_j}N\mathbb{1}_{[S_j\vee T_j,\infty)} - (\Delta_{S_j\vee T_j}N\mathbb{1}_{[S_j\vee T_j,\infty)})^{(p)}
$$

The process $(\Delta_{S_j\vee T_j}N\mathbb{1}_{[S_j\vee T_j,\infty)})^{(p)}$ is continuous, just because N has no jump at predictable times. Set $N'' = N - N'$ and compute the jump of N'' at $S_j \vee T_j$:

$$
\Delta_{S_j\vee T_j}N'' = \Delta_{S_j\vee T_j}N - \Delta_{S_j\vee T_j}N' = 0.
$$

This nullity entails that $(N'')_{S_j\vee T_j} \in \mathcal{G}^*_{S_j\vee T_j}$, so that $N''^{(S_j,T_j]} \in \mathcal{M}_{loc,0}(\mathbb{Q},\mathbb{G}^{(S_j,T_j]})$ (cf. Proposition 4.1 (5)). Because of the property $\mathfrak{Mrp}(\mathbb{Q},\mathbb{G}^{(S_j,T_j]},\widetilde{W}^{(S_j,T_j]})$, there exists a d-dimensional process $J \in \mathcal{I}(\mathbb{Q},\mathbb{G}^{(S_j,T_j]},\widetilde{W}^{(S_j,T_j]})$ such that $N''^{(S_j,T_j]} = J\boldsymbol{.}\widetilde{W}^{(S_j,T_j]}$. Applying Proposition 4.1 (6), Lemma 2.1, we have also $J\mathbb{1}_{(S_j,T_j]} \in \mathcal{I}(\mathbb{Q},\mathbb{G},\widetilde{W})$ and $N''^{(S_j,T_j]} = J\mathbb{1}_{(S_j,T_j]}\boldsymbol{.}\widetilde{W}$ in the sense of $(\mathbb{Q},\mathbb{G})$. On the other hand, we check immediately $[N''^{(S_j,T_j]}, N'^{(S_j,T_j]}] = 0$. These facts enable us to write

$$
[N''^{(S_j,T_j]}, N''^{(S_j,T_j]}] = [N, J\mathbb{1}_{(S_j,T_j]}\boldsymbol{.}\widetilde{W}_i] \in \mathcal{M}_{loc,0}(\mathbb{Q},\mathbb{G})
$$

This relation is possible only if $N''^{(S_j,T_j]} \equiv 0$, which yields $N^{(S_j,T_j]} = N'^{(S_j,T_j]}$. It follows that $\mathbb{1}_{\{S_j<t<T_j\}}\Delta_t N = 0$ and N has bounded variation on $(S_j,T_j]$

▶ Now, by covering condition $(\tau,\infty) = \cup_{j\in\mathbb{N}}(S_j,T_j)$, we conclude that N is a continuous local martingale with finite variation. It is therefore a constant, i.e., it is null. ∎

5 $\mathfrak{Mrp}$ on $\mathbb{R}_+$ and an equality between $\mathcal{G}_{\tau-}$ and $\mathcal{G}_\tau$

Putting together Theorem 3.2 and Theorem 4.2, we obtain immediately

Theorem 5.1 *Suppose Assumption 3.1. Suppose $\mathcal{H}$ measure condition covering (τ,∞). Then, $\mathfrak{Mrp}(\mathbb{Q},\mathbb{G},(\widetilde{W},L))$ and $\mathbb{1}_{\{0<\tau<\infty\}}W_\tau \in \mathcal{G}_{\tau-}$ hold, if and only if $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau = \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$.*

We see the particular role played by the equality $\{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-} = \{0 < \tau < \infty\} \cap \mathcal{G}_\tau$. In this section we show how this equality can be studied by $\mathcal{H}$ measure condition.

Theorem 5.2 *Suppose Assumption 3.1. Suppose*

*$\mathcal{H}$ **measure condition covering** $(0,\infty)$: There exists a countable family of $\mathbb{G}$-stopping times $\{S_j, T_j : j \in \mathbb{N}\}$, such that*

1. for any $j \in \mathbb{N}$, there exists a $\mathscr{H}$-measure $\mathbb{Q}_j$ over the time interval $(S_j, T_j]$.
2. T_j are $\mathbb{F}$-stopping times.
3. $(0, \infty) = \cup_{i \in \mathbb{N}}(S_j, T_j)$.

Suppose that τ avoids the $\mathbb{F}$ stopping times on $(0, \infty)$, i.e. $\mathbb{Q}[0 < \tau < \infty, \tau = T] = 0$ for every $\mathbb{F}$ stopping time T. Then, $\{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-} = \{0 < \tau < \infty\} \cap \mathcal{G}_\tau$.

Proof. Fixe $j \in \mathbb{N}$. Let ζ be an $\mathcal{F}_{T_j}$-measurable bounded random variable, and X be the martingale $X_t = \mathbb{Q}[\zeta|\mathcal{F}_t], 0 \leq t < \infty$. Let g be a bounded Borel function. We note the identity $\zeta = X_{T_j \vee t}$ for all $t \geq 0$. By definition of $\mathscr{H}$ measure, $X^{(S_j, T_j]}$ is a $(\mathbb{Q}_j, \mathbb{G}^{(S_j, T_j]})$ uniformly integrable martingale. Let $R_j = S_j \vee (\tau \wedge T_j)$. Note that $\{S_j < R_j < T_j\}$ is equivalent to $\{S_j < \tau < T_j\}$ (and in particular $R_j = \tau$). Applying Lemma A.5 and Proposition 4.1 (2), we can write

$$
\begin{aligned}
& \mathbb{1}_{\{S_j < \tau < T_j\}}\mathbb{Q}_j[g(\tau)\zeta|\mathcal{G}_\tau] \\
= \ & g(\tau)\mathbb{1}_{\{S_j < R_j < T_j\}}\mathbb{Q}_j[X^{S_j \vee T_j}_{S_j \vee T_j}|\mathcal{G}^{(S_j, T_j]}_{R_j}] \\
= \ & g(\tau)\mathbb{1}_{\{S_j < \tau < T_j\}}X_\tau \\
= \ & g(\tau)\mathbb{1}_{\{S_j < \tau < T_j\}}X_{\tau-} \quad \text{(see below for explication)} \\
\in \ & \mathcal{G}_{\tau-}
\end{aligned}
$$

Here the equality $\mathbb{1}_{\{S_j < \tau < T_j\}}X_\tau = \mathbb{1}_{\{S_j < \tau < T_j\}}X_{\tau-}$ holds because, firstly, τ avoids the $\mathbb{F}$ stopping times under $\mathbb{Q}$, and secondly, $\mathbb{Q}_j$ is equivalent to $\mathbb{Q}$ on $\{S_j < R_j < T_j\}$. Since $\{S_j < \tau < T_j\} \cap \mathcal{G}_\tau \subset \{S_j < \tau < T_j\} \cap (\sigma(\tau) \vee \mathcal{F}_{T_j})$, the above relation yields $\{S_j < \tau < T_j\} \cap \mathcal{G}_\tau = \{S_j < \tau < T_j\} \cap \mathcal{G}_{\tau-}$ under $\mathbb{Q}_j$, and hence, under $\mathbb{Q}$ by equivalence on $\{S_j < R_j < T_j\}$.

Now, for any $A \in \mathcal{G}_\tau$, noting that $(0, \infty) = \cup_{i \in \mathbb{N}}(S_j, T_j)$, under the probability $\mathbb{Q}$, we can write

$$
\begin{aligned}
\{0 < \tau < \infty\} \cap A \ = \ & \cup_{i \in \mathbb{N}}\{S_j < \tau < T_j\} \cap A \\
= \ & \cup_{i \in \mathbb{N}}\{S_j < \tau < T_j\} \cap B_j \quad \text{for some } B_j \in \mathcal{G}_{\tau-} \\
= \ & \cup_{i \in \mathbb{N}}\{S_j < \tau \leq T_j, \tau < \infty\} \cap B_j \quad \text{because } \tau \text{ avoids the } \mathbb{F} \text{ stopping times} \\
\in \ & \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}
\end{aligned}
$$

We have proved $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau \subset \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$ under $\mathbb{Q}$. But the inverse inclusion is an evidence. The theorem is proved. ∎

We end this section by the following relation between $\mathscr{H}$ measure condition covering $(0, \infty)$ and $\mathscr{H}$ measure condition covering (τ, ∞). It is the direct consequence of the definition.

Lemma 5.1 *If the family* $\{S_j, T_j : j \in \mathbb{N}\}$ *of* $\mathbb{G}$ *stopping times satisfies the* $\mathscr{H}$ *measure condition covering* $(0, \infty)$, *the family* $\{(S_j \vee \tau) \wedge (S_j \vee T_j), T_j : j \in \mathbb{N}\}$ *satisfies the* $\mathscr{H}$ *measure condition covering* (τ, ∞),

Proof. Consider four $\mathbb{G}$ stopping times S, T, U, V. Suppppose $S \leq U \leq S \vee T, S \leq V \leq S \vee T$. It can be checked that, for any process X, $(X^{(S,T]})^{(U,V]} = X^{(U,V]}$, and, by Proposition 4.1, Lemma A.5, for any $\mathbb{G}$ stopping time R such that $U \leq R \leq U \vee V$, $\mathcal{G}_R^{(U,V]} = \mathcal{G}_R^{(S,T]}$.

Let $\mathbb{Q}'$ be a $\mathscr{H}$ measure on $(S, T]$. Then, $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ on $\mathcal{G}_\infty$. For any bounded $(\mathbb{Q}, \mathbb{F})$ martingale X, $X^{(S,T]}$ is a $(\mathbb{Q}', \mathbb{G}^{(S, S \vee T]})$ local martingale. Consequently, $X^{(U,V]} = (X^{(S,T]})^{(U,V]}$ is a $(\mathbb{Q}', \mathbb{G}^{(S, S \vee T]})$ local martingale. Since $X^{(U,V]}$ is $\mathbb{G}^{(U, U \vee V]}$ adapted, for any $\mathbb{G}$ stopping times R, R' such that $U \leq R \leq R' \leq U \vee V$, for $A \in \mathcal{G}_R^{(U, U \vee V]}$, we have $A \in \mathcal{G}_R^{(S, S \vee T]}$ and therefore

$$
\begin{aligned}
\mathbb{Q}'[\mathbb{1}_A X_{R'}^{(U,V]}] \ = \ & \mathbb{Q}'[\mathbb{1}_A (X^{(S,T]})_{R'}^{(U,V]}] \\
= \ & \mathbb{Q}'[\mathbb{1}_A (X^{(S,T]})_R^{(U,V]}] \\
= \ & \mathbb{Q}'[\mathbb{1}_A X_R^{(U,V]}]
\end{aligned}
$$

i.e. $X^{(U,V]}$ is a $(\mathbb{Q}', \mathbb{G}^{(U, U \vee V]})$ local martingale. This achieves the proof of the lemma. ∎

6 Examples

In this section we assume the setting of subsection 1.1. We show that our method applies in all the cases studied in the literature. We have now a uniform way to prove various classical results. In all this section, Assumption 3.1 (i) is in force.

6.1 The case of the immersion condition

Suppose the immersion condition ([4, 6, 26]), i.e., any $(\mathbb{Q}, \mathbb{F})$ local martingale is a $(\mathbb{Q}, \mathbb{G})$ local martingale (in particular $W = \widetilde{W}$). In this case, if we take $T = \infty$ and $S = 0$, the random variable T is a $\mathbb{F}$ stopping time and the interval $(S, T]$ covers $(0, \infty)$, and the probability measure $\mathbb{Q}$ is clearly an $\mathscr{H}$-measure on $(S, T]$. Hence, according to Lemma 5.1 and Theorem 5.1, the properties $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}, (W, L))$ and $\mathbb{1}_{\{0 < \tau < \infty\}} W_\tau \in \mathcal{F}_{\tau-}$ hold, whenever $\{0 < \tau < \infty\} \cap \mathcal{G}_\tau = \{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-}$.

Let us show that the immersion condition together with $W_\tau \in \mathcal{F}_{\tau-}$ implies $\mathcal{G}_{\tau-} = \mathcal{G}_\tau$. Let ζ be a bounded $\mathcal{F}_\infty$ measurable random variable. Let $\zeta_t = \mathbb{Q}[\zeta | \mathcal{F}_t], t \geq 0$. Since $W_\tau \in \mathcal{F}_{\tau-}$, by the property $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{F}, W)$, we check that $\Delta_\tau \zeta \in \mathcal{F}_{\tau-}$ so that $\zeta_\tau \in \mathcal{F}_{\tau-} = \mathcal{G}_{\tau-}$ (see the proof of Theorem 3.2). Thanks to the immersion condition, if g is a bounded Borel function on $[0, \infty]$,

$$\mathbb{Q}[g(\tau)\zeta | \mathcal{G}_\tau] = g(\tau)\mathbb{Q}[\zeta | \mathcal{G}_\tau] = g(\tau)\zeta_\tau \in \mathcal{G}_{\tau-}.$$

Applying the monotone class theorem, we obtain $\mathcal{G}_\tau = \mathcal{G}_{\tau-}$.

Theorem 6.1 *Suppose Assumption 3.1 (i). Then the following two conditions are equivalent*

   (i) Immersion condition and $W_\tau \in \mathcal{F}_{\tau-}$

  (ii) $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}, (W, L))$ and $\mathcal{G}_\tau = \mathcal{G}_{\tau-}$.

Remark 6.1 Theorem 6.1 implies the results given in [26, Theorem 2.3, $N = 1$], because, for W a Brownian motion, $W_\tau \in \mathcal{F}_{\tau-}$. Applying Theorem 6.1 and Lemma 2.4, we can prove also [26, Theorem 3.2, $N = 1$].

6.2 The case of a honest time

In this section we suppose that τ is an $\mathbb{F}$ honest time. Honest time has been fully studied in the past (cf. [1, 21, 24]). We reconsider this case as an example of application of our results.

To simplify the computations, we assume

Hy(C) All $(\mathbb{Q}, \mathbb{F})$ local martingales are continuous.

We know that, when τ is a honest time, $(\mathcal{H}')$ hypothesis holds. Under **Hy**(C), $\hat{A} - A \equiv 0$ (see Section 1.1 for notations) and the drift operator $\Gamma(X)$ on (τ, ∞) is given by

$$\mathbb{1}_{(\tau, \infty)} \cdot \Gamma(X) = \mathbb{1}_{(\tau, \infty)} \frac{1}{1 - Z} \cdot \langle M, X \rangle$$

which is continuous.

Lemma 6.1 *Suppose Assumption 3.1 (i) and **Hy**(C). We have the $\mathscr{H}$ measure condition covering (τ, ∞).*

Proof. For any $n \in \mathbb{N}^*$, for any $a > 0$, set

$$T_{a,n} = \inf\{t \geq a : \int_a^t \frac{1}{(1 - Z_{s-})^2} d\langle M \rangle_s > n\} \wedge n.$$

The random variables $T_{a,n}$ are $\mathbb{F}$ stopping times. Set $S_a = \tau \vee a$. Since $Z_{t-} < 1, Z_t < 1$, for $t > \tau$, we have $\cup_{a \in \mathbb{Q}, a>0, n \in \mathbb{N}^*}(S_a, T_{a,n}) = (\tau, \infty)$. For fixed $a > 0, n \in \mathbb{N}^*$, we introduce the process

$$\eta = \mathcal{E}\left(\mathbb{1}_{(S_a, T_{a,n}]} \frac{1}{1 - Z_-} \cdot \widetilde{M}\right)$$

which is a positive continuous $(\mathbb{Q}, \mathbb{G})$ martingale. We now show that the probability measure

$$\mathbb{Q}^{a,n} = \eta_{S_a \vee T_{a,n}} \cdot \mathbb{Q}$$

is a $\mathscr{H}$-measure on the random interval $(S_a, T_{a,n}]$.

Let X be a $(\mathbb{Q}, \mathbb{F})$ local martingale. Then, $\widetilde{X}$ is a $(\mathbb{Q}, \mathbb{G})$ local martingale. By Girsanov theorem, the process $\widetilde{X}^{[\eta]}$ (cf. Lemma 2.4) is a $\mathbb{Q}^{a,n}$-$\mathbb{G}$ local martingale. Let us compute $\widetilde{X}^{[\eta]}$ on the random interval $(S_a, T_{a,n}]$. Thanks to **Hy**(C), $\langle \widetilde{X}, \widetilde{M} \rangle = \langle X, M \rangle$, $\langle \eta, \widetilde{X} \rangle = \langle \eta, X \rangle$.

$$\widetilde{X}^{[\eta]}_{\tau \vee t} - \widetilde{X}^{[\eta]}_{\tau \vee a}$$
$$= X_{\tau \vee t} - X_{\tau \vee a} + \int_a^t \mathbb{1}_{\{\tau < s\}} \frac{1}{1 - Z_{s-}} d\langle M, X \rangle_s - \int_a^t \mathbb{1}_{\{\tau < s\}} \frac{1}{1 - Z_{s-}} \mathbb{1}_{\{S_a < s \leq T_{a,n}\}} d\langle \widetilde{M}, X \rangle_s$$

$$= X_{\tau \vee t} - X_{\tau \vee a}, \quad \text{if } a \leq t \leq T_n$$

It follows that $X^{\tau \vee a \vee T_{a,n}}_{\tau \vee a \vee t} - X_{\tau \vee a}, t \geq 0$, is a $(\mathbb{Q}^{a,n}, \mathbb{G})$ local martingale. Since $X_{\tau \vee a \vee T_{a,n}} \in \mathcal{F}_{\tau \vee a \vee T_{a,n}} \subset \mathcal{G}^*_{\tau \vee a \vee T_{a,n}}$, we conclude that $X^{(S_a, T_{a,n}]}$ is a $(\mathbb{Q}^{a,n}, \mathbb{G}^{(S_a, T_{a,n}]})$ local martingale (cf. Proposition 4.1 (5)). This proves that the probability $\mathbb{Q}^{a,n}$ is an $\mathscr{H}$-measure on $(S_a, T_{a,n}]$. ■

Now we apply Theorem 3.1 and Theorem 4.2. Note that, according to [21, Proposition(5.3)], for any $\mathbb{G}$-predictable process J, there exist $\mathbb{F}$-predictable processes J', J'' such that

$$J \mathbb{1}_{(0,\infty)} = J' \mathbb{1}_{(0,\tau]} + J'' \mathbb{1}_{(\tau,\infty)}$$

Theorem 6.2 *Suppose Assumption 3.1 (i) and **Hy**(C). For any bounded $(\mathbb{Q}, \mathbb{G})$-martingale, there exist $\mathbb{F}$-predictable processes J', J'', K and a bounded $\xi \in \mathcal{G}_\tau$ such that $\mathbb{Q}[\xi | \mathcal{G}_{\tau-}] = 0$, $J' \mathbb{1}_{(0,\tau]} + J'' \mathbb{1}_{(\tau,\infty)} \in \mathcal{I}(\mathbb{Q}, \mathbb{G}, \widetilde{W})$ and, for $t \geq 0$,*

$$X_t = X_0 + \int_0^t \left(J' \mathbb{1}_{\{s \leq \tau\}} + J'' \mathbb{1}_{\{\tau < s\}}\right) d\widetilde{W}_s + \int_0^t K_s \mathbb{1}_{\{0 < s \leq \tau\}} dL_s + \mathbb{1}_{\{\tau > 0\}} \xi H_t, \tag{14}$$

If, in addition, $\{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-} = \{0 < \tau < \infty\} \cap \mathcal{G}_\tau$, the property $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}, (\widetilde{W}, L))$ holds.

We end this section by a remark on Brownian filtrations (in the sense of [2])

Theorem 6.3 *Suppose Assumption 3.1 (i). Suppose that $\mathbb{F}$ is a Brownian filtration. Then there exists a bounded $(\mathbb{Q}, \mathbb{G})$ martingale ν such that $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}, (\widetilde{W}, L, \nu))$ holds.*

Proof. According to [2], since $\mathbb{F}$ is a brownian filtration and τ is $\mathbb{F}$ honest, there exists a random event $A \in \mathcal{G}_\tau$ such that $\mathcal{G}_\tau = \mathcal{G}_{\tau-} \vee \sigma(A)$. This means that, for any $\xi \in \mathcal{G}_\tau$, there exist $\xi', \xi'' \in \mathcal{G}_{\tau-}$ such that $\xi = \xi' \mathbb{1}_A + \xi'' \mathbb{1}_{A^c}$. In particular, if $\mathbb{Q}[\xi | \mathcal{G}_{\tau-}] = 0$, i.e., $0 = \xi' p + \xi''(1 - p)$, where $p = \mathbb{Q}[A | \mathcal{G}_{\tau-}]$, we have

$$\xi = (-\mathbb{1}_{\{p>0\}} \xi'' \frac{1}{p} + \mathbb{1}_{\{p=0\}} \xi' \frac{1}{1-p})((1 - p) \mathbb{1}_A - p \mathbb{1}_{A^c})$$

Let $\nu = ((1-p)\mathbb{1}_A - p\mathbb{1}_{A^c})H$ and F be a $\mathbb{F}$-predictable process such that $F_\tau = (-\mathbb{1}_{\{p>0\}}\xi''\frac{1}{p} + \mathbb{1}_{\{p=0\}}\xi'\frac{1}{1-p})$. ν is a bounded $(\mathbb{Q}, \mathbb{G})$ martingale and F is ν-integrable. For bounded $(\mathbb{Q}, \mathbb{G})$-martingale X, the formula (14) now becomes

$$X_t = X_0 + \int_0^t \left(J'\mathbb{1}_{\{s \leq \tau\}} + J''\mathbb{1}_{\{\tau < s\}} \right) d\widetilde{W}_s + \int_0^t K_s \mathbb{1}_{\{0 < s \leq \tau\}} dL_s + \int_0^t F_s d\nu_s$$

This proves the theorem. ∎

6.3 The case of density hypothesis

In this subsection we work under

Assumption 6.1 *Density Hypothesis* *We assume that, for any $t \in \mathbb{R}_+$, there exists a strictly positive $\mathcal{B}[0,\infty] \otimes \mathcal{F}_t$ measurable function $\alpha_t(\theta, \omega), (\theta, \omega) \in [0,\infty] \times \Omega$, which gives the conditional law*

$$\mathbb{Q}[\tau \in \mathsf{A}|\mathcal{F}_t] = \int_{\mathsf{A}} \alpha_t(\theta)\mu(d\theta), \ t \geq 0, \mathsf{A} \in \mathcal{B}[0,\infty]$$

where $\alpha_t(\theta)$ denotes the application $\alpha_t(\theta, \cdot)$ and μ is a diffuse probability measure on $\mathbb{R}_+$. We assume that the trajectory $t \to \alpha_t(\theta, \omega)$ is càdlàg.

For any $n \geq 1$, let $\mathbb{Q}'_n = \alpha_n(\tau)^{-1} \cdot \mathbb{Q}$. We check that $\mathbb{Q}'_n[\tau \in d\theta|\mathcal{F}_n] = \mu(d\theta)$. This means that, under $\mathbb{Q}'_n$, τ is independent of $\mathcal{F}_n$. For any $(\mathbb{Q}'_n, \mathbb{F})$-local martingale Y, Y^n (Y stopped at n) will be a $(\mathbb{Q}'_n, \mathbb{G})$-local martingale, and therefore $Y^{(0,n]}$ is a $(\mathbb{Q}'_n, \mathbb{G}^{(0,n]})$-local martingale. Since

$$\mathbb{Q}[\frac{1}{\alpha_n(\tau)}|\mathcal{F}_n] = 1.$$

we have $\mathbb{Q}'_n|_{\mathcal{F}_n} = \mathbb{Q}|_{\mathcal{F}_n}$. This yields that, for any $(\mathbb{Q}, \mathbb{F})$-local martingale X, X^n will be a $(\mathbb{Q}'_n, \mathbb{F})$-local martingale, and $X^{(0,n]}$ is a $(\mathbb{Q}'_n, \mathbb{G}^{(0,n]})$-local martingale. We have just proved that $\mathbb{Q}'_n$ is a $\mathscr{H}$-measure on $(0,n]$. We note that the integers n are $\mathbb{F}$ stopping times and the interval $(0,n], n \geq 1$, covers $(0,\infty)$. Note also that, since μ is diffuse, τ avoids the $\mathbb{F}$ stopping times.

Applying Theorem 5.2, Lemma 5.1, Theorem 5.1, we obtain :

Theorem 6.4 *Suppose Assumption 3.1 (i), and Density hypothesis 6.1. Then, $\{0 < \tau < \infty\} \cap \mathcal{G}_{\tau-} = \{0 < \tau < \infty\} \cap \mathcal{G}_\tau$ and $\mathfrak{Mrp}(\mathbb{Q}, \mathbb{G}, (\widetilde{W}, L))$ holds.*

Remark 6.2 The property $\mathfrak{Mrp}$ under Density hypothesis 6.1 has been studied in [17, Theorem 2.1] when W is continuous. The method used in [17] is Itô's computations.

6.4 The case of Cox measure and the related ones

In this subsection, we consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with an filtration $\mathbb{F}$ of sub-σ-algebras in $\mathcal{A}$. We consider the product measurable space $([0,\infty] \times \Omega, \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty)$. As usual, we consider $\mathbb{F}$ as an filtration on the product space and $\mathbb{P}$ as a probability measure defined on $\mathcal{F}_\infty$ considered as a sub-σ-algebra of $\mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty$ (see Section 7). Consider the projection map : $\tau(s, \omega) = s$ for $(s, \omega) \in [0,\infty] \times \Omega$. Let Λ be a continuous increasing $\mathbb{F}$-adapted process such that $\Lambda_0 = 0, \Lambda_\infty = \infty$. The Cox measure ν^Λ on the product space $[0,\infty] \times \Omega$ associated with Λ is defined by the relation

$$\nu^\Lambda[A \cap \{s < \tau \leq t\}] = \mathbb{P}[\mathbb{1}_A \int_s^t e^{-\Lambda_v} d\Lambda_v], \ A \in \mathcal{F}_\infty, \ 0 < s < t < \infty \tag{15}$$

Consider the progressively enlarged filtration $\mathbb{G}$ on the product space $[0,\infty] \times \Omega$ from $\mathbb{F}$ with τ. It is well know that, under the Cox measure, the immersion condition holds (cf. [4]). It is also easy to check that $\nu^\Lambda[\tau = T] = 0$ for any $\mathbb{F}$-stopping time T, consequence of the continuity of the process Λ. This last property implies $W_\tau \in \mathcal{F}_{\tau-}$. Theorem 6.1 is applicable. We have the property $\mathfrak{M}rp(\nu^\Lambda, \mathbb{G}, (W, H - \frac{1}{1-e^{-\Lambda}} \cdot \Lambda))$ and $\mathcal{G}_{\tau-} = \mathcal{G}_\tau$.

Now, if a probability measure $\mathbb{Q}$ on the product space is absolutely continuous with respect to the Cox measure, we apply Lemma 2.4 to obtain

Theorem 6.5 *Suppose Assumption 3.1 (i). If the probability measure $\mathbb{Q}$ is absolutely continuous with respect to the Cox measure ν^Λ, we have the properties $W_\tau \in \mathcal{F}_{\tau-}$ and $\mathcal{G}_{\tau-} = \mathcal{G}_\tau$ and $\mathfrak{M}rp(\mathbb{Q}, \mathbb{G}, (\widetilde{W}, L))$.*

Remark 6.3 It is proved in [18] that, for any probability measure $\mathbb{P}$ on $\mathcal{F}_\infty$, for any positive $(\mathbb{P}, \mathbb{F})$ local martingale N, for any continuous $\mathbb{F}$-adapted increasing process Λ such that $\Lambda_0 = 0$, $N_0 = 1$ and $\forall t > 0, N_{t\pm}e^{-\Lambda} < 1$, there exists always a probability measures $\mathbb{Q}$ on the product space, which is absolutely continuous with respect to the Cox measure, such that $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$ and $\mathbb{Q}[t < \tau|\mathcal{F}_\infty] = N_t e^{-\Lambda_t}, t \geq 0$.

7   ♮-model

In this section, we consider the ♮-model developed in [19]. The basic setting is a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual condition. We consider an $\mathbb{F}$-adapted continuous increasing process Λ and a càdlàg positive $(\mathbb{P}, \mathbb{F})$ local martingale N such that $\Lambda_0 = 0, N_0 = 1$ and $0 \leq N_t e^{-\Lambda_t} \leq 1$ for all $0 \leq t < \infty$. We consider the product measurable space $([0,\infty] \times \Omega, \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty)$ with its canonical projection maps π and $\tau : \pi(s,\omega) = \omega$ and $\tau(s,\omega) = s$. Using the projection map π we pull back the probability structure $(\mathbb{P}, \mathbb{F})$ onto the product space $[0,\infty] \times \Omega$ with the filtration $\hat{\mathbb{F}} = \pi^{-1}(\mathbb{F})$ and with the probability measure on $\pi^{-1}(\mathcal{F}_\infty)$ defined by $\hat{\mathbb{P}}(\pi^{-1}(A)) = \mathbb{P}(A)$ for $A \in \mathcal{F}_\infty$. The probability structure $([0,\infty] \times \Omega, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ is isomorphic to that of $(\Omega, \mathbb{F}, \mathbb{P})$. We will henceforth simply denote $(\hat{\mathbb{P}}, \hat{\mathbb{F}})$ by $(\mathbb{P}, \mathbb{F})$ and identify the $\mathcal{F}_\infty$-measurable random variables ξ on Ω with $\xi \circ \pi$ on the product space.

We consider the following problem :

Problem $\mathcal{P}^*$. Construct on the product space $([0,\infty] \times \Omega, \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty)$ a probability measure $\mathbb{Q}$ such that
– (restriction condition) $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$ and
– (projection condition) $\mathbb{Q}[\tau > t|\mathcal{F}_t] = N_t e^{-\Lambda_t}$ for all $0 \leq t < \infty$.
(Recall that we identify $\hat{\mathbb{F}}$ as $\mathbb{F}$ and $\hat{\mathbb{P}}$ as $\mathbb{P}$.)

Suppose $\mathbf{Hy}(C)$, i.e. all $(\mathbb{P}, \mathbb{F})$ local martingales are continuous. Suppose $Z_t < 1$ for any $0 < t < \infty$, where $Z = Ne^{-\Lambda}$. Under these conditions, [19] proves that there exist infinity of solutions to the problem $\mathcal{P}^*$. In particular, for any $(\mathbb{P}, \mathbb{F})$ local martingale Y, for any bounded differentiable function f with bounded continuous derivative and $f(0) = 0$, there exists $\mathbb{Q}^\natural$ a solution of the problem $\mathcal{P}^*$ on the product space such that, for any $u \in \mathbb{R}_+^*$, the martingale $M_t^u = \mathbb{Q}^\natural[\tau \leq u|\mathcal{F}_t], t \geq u$, satisfies the following equation(♮)

$$(\natural_u) \begin{cases} dX_t = X_t \left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(X_t - (1 - Z_t))dY_t \right), & u \leq t < \infty \\ X_u = 1 - Z_u \end{cases}$$

Consider the progressively enlarged filtration $\mathbb{G}$ on this product space with the random time τ.

Theorem 7.1 *Suppose the same assumptions as above. Suppose in addition that*

Hy(Mc) : *For each $0 < t < \infty$, the map $u \to M_t^u$ is continuous on $(0,t]$.*

Then, for any $(\mathbb{P}, \mathbb{F})$ local martingale X, the process

$$\Gamma(X)_t := \int_0^t \mathbb{1}_{\{s \leq \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X\rangle_s + \int_0^t \mathbb{1}_{\{\tau < s\}} \left(-\frac{e^{-\Lambda_s}}{1 - Z_s}\right) d\langle N, X\rangle_s$$

$$+ \int_0^t \mathbb{1}_{\{\tau < s\}} (f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s))) d\langle Y, X\rangle_s, \quad 0 \leq t < \infty,$$

is a well-defined $\mathbb{G}$-predictable process with finite variation, and the process $\widetilde{X} = X - \Gamma(X)$ is a $(\mathbb{Q}^\natural, \mathbb{G})$ local martingale.

We now study the $\mathfrak{Mrp}$ property for the $\natural$-model. Let W be a càdlàg d-dimensional $\mathbb{F}$-adapted process. We assume the following set of assumptions :

Assumption 7.1 (i) *The above two parameters Y and f are given.*
(ii) *Assume **Hy**(C), **Hy**(Mc)*
(iii) *Assume $\mathfrak{Mrp}(\mathbb{P}, \mathbb{F}, W)$ and the ellipticity assumption for W under $\mathbb{Q}^\natural$.*
(iv) *Assume $0 < Z_t < 1$ for $0 < t < \infty$,*

For any $\mathbb{F}$ stopping time, under assumption **Hy**(Mc), we have

$$\mathbb{Q}^\natural[\tau = T, \tau \leq t | \mathcal{F}_t] = \int_0^t \mathbb{1}_{\{u = T\}} d_u M_t^u = 0, \ \forall 0 < t < \infty$$

This yields that τ avoids the $\mathbb{F}$ stopping times on $(0, \infty)$.

Let $0 < a < \infty, n \in \mathbb{N}^*$ and let

$$
\begin{aligned}
T_{a,n} \ &= \inf\{v \geq a : \quad \int_a^v \frac{e^{-2\Lambda_w}}{Z_w^2} d\langle N\rangle_w > n, \\
&\qquad\qquad \text{or } \int_a^v \frac{e^{-2\Lambda_w}}{(1 - Z_w)^2} d\langle N\rangle_w > n, \\
&\qquad\qquad \text{or } \langle Y\rangle_v - \langle Y\rangle_a > n, \\
&\qquad\qquad \text{or } \langle W\rangle_v - \langle W\rangle_a > n, \\
&\qquad\qquad \text{or } v > a + n \qquad\qquad \}.
\end{aligned}
$$

$T_{a,n}$ is a $\mathbb{F}$-stopping time and, since $0 < Z < 1$ on $(0, \infty)$, since N, Y, W are continuous, $\lim_{n \to \infty} T_{a,n} = \infty$. We have $(0, \infty) = \cup_{a \in \mathbb{Q}, n \in \mathbb{N}^*} (a, T_{a,n})$.

Let us show that there exists a $\mathscr{H}$-measure on the intervals $(a, T_{a,n}]$. We introduce

$$\gamma_s = \frac{e^{-\Lambda_s}}{Z_s}, \quad \alpha_s = -\frac{e^{-\Lambda_s}}{1 - Z_s}, \quad \beta_s = f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s))$$

and the exponential martingale :

$$\eta = \mathcal{E}\left((-\gamma \mathbb{1}_{[0,\tau]} - \alpha \mathbb{1}_{(\tau,\infty)}) \mathbb{1}_{(a,T_n]} \bullet \widetilde{N} + (-\beta) \mathbb{1}_{(\tau,\infty)} \mathbb{1}_{(a,T_n]} \bullet \widetilde{Y}\right)$$

By **Hy**(C), $\Gamma(N)$ is continuous and $\langle N\rangle = [N] = \langle \widetilde{N}\rangle$ indifferently in the filtration $\mathbb{F}$ or in the filtration $\mathbb{G}$. The same property holds for the bracket of Y. We check then that Novikov's condition is satisfied by η so that $\mathbb{Q}^\natural[\eta] = 1$. Let $\mathbb{Q}^{a,n} = \eta \cdot \mathbb{Q}^\natural$. By Girsanov theorem, the process $\widetilde{X}_t^{[\eta]} - \widetilde{X}_a^{[\eta]}, \ 0 \leq t < \infty$, is a $\mathbb{Q}^{a,n}$-$\mathbb{G}$ local martingale. Note

$$\Gamma(X) = \gamma \mathbb{1}_{[0,\tau]} \bullet \langle N, X\rangle + \alpha \mathbb{1}_{(\tau,\infty)} \bullet \langle N, X\rangle + \beta \mathbb{1}_{(\tau,\infty)} \bullet \langle Y, X\rangle$$

Because of **Hy**(C), we can write $\langle \widetilde{X}, \widetilde{N}\rangle = \langle X, N\rangle$ and $\langle \widetilde{X}, \widetilde{Y}\rangle = \langle X, Y\rangle$, and therefore, by a direct computation (cf. subsection 6.2), we get

$$\widetilde{X}_t^{[\eta]} - \widetilde{X}_a^{[\eta]} = X_t - X_a, \quad \text{if } a \leq t \leq T_{a,n}$$

This shows that $X_{a\vee t}^{T_{a,n}} - X_a$ is a $(\mathbb{Q}^{a,n}, \mathbb{G})$ local martingale. Since $X_{T_{a,n}} - X_a \in \mathcal{F}_{T_{a,n}} \subset \mathcal{G}_{T_{a,n}}^*$, $X^{(a,T_{a,n}]}$ is also a $(\mathbb{Q}^{a,n}, \mathbb{G}^{(a,T_{a,n}]})$ local martingale (cf. Proposition 4.1 (5)). The measure $\mathbb{Q}^{a,n}$ is an $\mathcal{H}$-measure on $(a, T_{a,n}]$.

The $\mathcal{H}$-measure condition covering $(0, \infty)$ is satisfied. Applying Theorem 5.2, Lemma 5.1, Theorem 5.1, we obtain

Theorem 7.2 *Under Assumption 7.1, the property* $\mathfrak{M}rp(\mathbb{Q}^\natural, \mathbb{G}, (\widetilde{W}, L))$ *holds.*

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A   Study of the filtration $\mathbb{G}^{(S,T]}$

In this appendix, we study the filtrations $\mathbb{G}^{(S,T]}$ introduced in Section 4.1 and prove the results stated in Proposition 4.1. This study is independent of the main text of this article. We consider a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ of sub-σ-algebras in $\mathcal{A}$. We consider a random time τ in $\mathcal{A}$ and its associated progressively enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ where $\mathcal{G}_t = \cap_{s>t}(\mathcal{F}_s \vee \sigma(\tau \wedge s))$ for $t \in \mathbb{R}_+$. Unlike the assumptions in Subsection 1.1, we do not assume here that $\mathcal{G}_0$ contains all the $(\mathbb{Q}, \mathcal{G}_\infty)$ negligible sets.

We recall two useful results :

Lemma A.1 *Let E be a space. Let $\mathcal{C}$ be a non empty family of sets in E. Let $A \subset E$. Then, $A \cap \sigma(\mathcal{C}) = A \cap \sigma(A \cap \mathcal{C})$.*

Lemma A.2 *(cf. [21, Lemme (4.4)]) Let T be a $\mathbb{G}$-stopping time. We have*

$$
\begin{aligned}
\mathcal{G}_{T-} &= \{T \leq \tau, T < \infty\} \cap \mathcal{F}_{T-} + \{\tau < T < \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_{T-}) + \{T = \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_\infty) \\
&= \{T < \tau\} \cap \mathcal{F}_{T-} + \{\tau \leq T < \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_{T-}) + \{T = \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_\infty)
\end{aligned}
$$

From now on till Lemma A.11, we consider two $\mathbb{G}$-stopping times S, T such that $S \leq T$.

Lemma A.3 *We have the relations :* $\mathcal{G}_\infty^{(S,T]} = \mathcal{G}_T^* \subset \sigma(\tau) \vee \mathcal{F}_T$ *and* $\mathcal{G}_{T-} \subset \mathcal{G}_T^* \subset \mathcal{G}_T$. *For* $t \geq 0$, *we have* $\mathcal{G}_t^{(S,T]} \subset \mathcal{G}_{(S\vee t)\wedge T}$. *A random variable* ξ *is* $\mathcal{G}_t^{(S,T]}$-*measurable if and only if there exist two random variables* $\xi_1 \in \mathcal{G}_T^*$ *and* $\xi_2 \in \mathcal{G}_{S\vee t}$ *such that*

$$\xi = \xi_1 \mathbb{1}_{\{T \leq S\vee t\}} + \xi_2 \mathbb{1}_{\{S\vee t < T\}}$$

or if and only if $\xi\mathbb{1}_{\{T \leq S\vee t\}} \in \mathcal{G}_T^*$ *and* $\xi\mathbb{1}_{\{S\vee t < T\}} \in \mathcal{G}_{S\vee t}$.

Proof. The results of the lemma are direct consequences of the definition. Let us prove, for example, the relation $\mathcal{G}_T^* \subset \sigma(\tau) \vee \mathcal{F}_T$. We note first that $\mathbb{1}_{[0,t]}(T) \in \mathcal{F}_{T-} \subset \mathcal{F}_T$, and consequently,

$$\{T < \tau\} = \cup_{t\in\mathbb{Q}_+}\{T \leq t\} \cap \{t < \tau\} \in \sigma(\tau) \vee \mathcal{F}_T$$

In the same way, we can prove that $\{\tau \leq T\}$ and $\{T = \infty\}$ belong to $\sigma(\tau) \vee \mathcal{F}_T$. Therefore,

$$\begin{aligned}
\mathcal{G}_T^* &\subset \{T < \tau\} \cap (\sigma(\tau) \vee \mathcal{F}_T) + \{\tau \leq T < \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_T) + \{T = \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_T) \\
&= \sigma(\tau) \vee \mathcal{F}_T
\end{aligned}$$

where we have used the relation $\{T = \infty\} \cap \mathcal{F}_T = \{T = \infty\} \cap \mathcal{F}_\infty$. The lemma is proved. ∎

Lemma A.4 *(Proposition 4.1 (1))* $\mathbb{G}^{(S,T]}$ *is a right-continuous filtration.*

Proof. Let us show first that $\mathbb{G}^{(S,T]}$ is a filtration, i.e., $\mathcal{G}_s^{(S,T]} \subset \mathcal{G}_t^{(S,T]}$ for any $0 < s < t < \infty$. By definition,

$$\begin{aligned}
&\mathcal{G}_t^{(S,T]} \\
=\ & \{\{T \leq S \vee t\} \cap (\{T \leq S \vee s\} \cap A + \{S \vee s < T\} \cap A) + \{S \vee t < T\} \cap B : A \in \mathcal{G}_T^*,\ B \in \mathcal{G}_{S\vee t}\} \\
=\ & \{\{T \leq S \vee s\} \cap A + \{S \vee s < T\} \cap (\{T \leq S \vee t\} \cap A' + \{S \vee t < T\} \cap B) : A, A' \in \mathcal{G}_T^*,\ B \in \mathcal{G}_{S\vee t}\}
\end{aligned}$$

We study $\mathcal{G}^{(S,T]}$ separately in two cases : $\{t \leq S\}$ or $\{S < t\}$. Firstly, we have

$$\begin{aligned}
&\{t \leq S\} \cap \mathcal{G}_t^{(S,T]} \\
=\ & \{\{t \leq S\} \cap \{T \leq S\} \cap A + \{t \leq S\} \cap \{S < T\} \cap B : A \in \mathcal{G}_T^*,\ B \in \mathcal{G}_S\} \\
=\ & \{\{t \leq S\} \cap \{T \leq S \vee s\} \cap A + \{t \leq S\} \cap \{S \vee s < T\} \cap B : A \in \mathcal{G}_T^*,\ B \in \mathcal{G}_{S\vee s}\} \\
=\ & \{t \leq S\} \cap \mathcal{G}_s^{(S,T]}
\end{aligned}$$

Next, we note that, on the set $\{S < t\}$, we have $S \vee s < t$. Therefore

$$\begin{aligned}
&\{S < t\} \cap \mathcal{G}_t^{(S,T]} \\
=\ & \{\{S < t\} \cap \{T \leq S \vee s\} \cap A + \{S < t\} \cap \{S \vee s < T\} \cap (\{T \leq t\} \cap A' + \{t < T\} \cap B) \\
&\hspace{6cm} : A, A' \in \mathcal{G}_T^*,\ B \in \mathcal{G}_{S\vee t}\} \\
\supset\ & \{\{S < t\} \cap \{T \leq S \vee s\} \cap A + \{S < t\} \cap \{S \vee s < T\} \cap (\{T \leq t\} \cap A' + \{t < T\} \cap B) \\
&\hspace{6cm} : A \in \mathcal{G}_T^*,\ A' \in \mathcal{G}_{T-},\ B \in \mathcal{G}_{t-}\} \\
&\quad \text{because, from Lemma A.3, } \mathcal{G}_{T-} \subset \mathcal{G}_T^* \\
\\
=\ & \{\{S < t\} \cap \{T \leq S \vee s\} \cap A + \{S < t\} \cap \{S \vee s < T\} \cap C : A \in \mathcal{G}_T^*,\ C \in \mathcal{G}_{(T\wedge t)-}\} \\
&\quad \text{see [13, Corollary 3.5]} \\
\supset\ & \{\{S < t\} \cap \{T \leq S \vee s\} \cap A + \{S < t\} \cap \{S \vee s < T \wedge t\} \cap C : A \in \mathcal{G}_T^*,\ C \in \mathcal{G}_{S\vee s}\} \\
=\ & \{S < t\} \cap \mathcal{G}_s^{(S,T]}
\end{aligned}$$

This shows that $\mathbb{G}^{(S,T]}$ is a filtration.

Now we show the right-continuity, i.e., $\mathcal{G}_s^{(S,T]} = \cap_{t>s}\mathcal{G}_t^{(S,T]}$ for any $s \in \mathbb{R}_+$. We begin with the following observation. Let D be a set in Ω. We have the equality

$$D \cap (\cap_{t\in\mathbb{Q},t>s}\mathcal{G}_t^{(S,T]}) = \cap_{t\in\mathbb{Q},t>s}(D \cap \mathcal{G}_t^{(S,T]}) \tag{16}$$

Clearly the left-hand side family is contained in the right one. Let B be an element in the right-hand side family. For any $t \in \mathbb{Q}$ with $t > s$, there exists $B_t \in \mathcal{G}_t^{(S,T]}$ such that $B = D \cap B_t$. It yields that, for $\beta > 0$, $B = \cup_{t \in \mathbb{Q}, \beta > t > s}(D \cap B_t) = D \cap (\cup_{t \in \mathbb{Q}, \beta > t > s} B_t)$ and

$$B = \cap_{\beta > s} \cup_{t \in \mathbb{Q}, \beta > t > s} (D \cap B_t) = D \cap (\cap_{\beta > s} \cup_{t \in \mathbb{Q}, \beta > t > s} B_t)$$

By monotonicity

$$\cap_{\beta > s} \cup_{t \in \mathbb{Q}, \beta > t > s} B_t = \cap_{s + \epsilon > \beta > s} \cup_{t \in \mathbb{Q}, \beta > t > s} B_t \in \mathcal{G}_{s+\epsilon}^{(S,T]}$$

for any $\epsilon > 0$. This shows that B is an element in the left hand side family. The formula (16) is proved. In the same way we prove also the equality

$$D \cap (\cap_{t \in \mathbb{Q}, t > s} \mathcal{G}_{S \vee t}) = \cap_{t \in \mathbb{Q}, t > s}(D \cap \mathcal{G}_{S \vee t})$$

Using the above observation, we obtain, on the one hand,

$$
\begin{aligned}
\{T \leq S \vee s\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) &= \cap_{t \in \mathbb{Q}, t > s}(\{T \leq S \vee s\} \cap \mathcal{G}_t^{(S,T]}) \\
&= \{T \leq S \vee s\} \cap \mathcal{G}_T^* \\
&= \{T \leq S \vee s\} \cap \mathcal{G}_s^{(S,T]}
\end{aligned}
$$

and on the other hand, for $\beta > s$,

$$
\begin{aligned}
\{S \vee \beta < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) &= \cap_{t \in \mathbb{Q}, \beta > t > s}(\{S \vee \beta < T\} \cap \mathcal{G}_t^{(S,T]}) \\
&= \{S \vee \beta < T\} \cap (\cap_{t \in \mathbb{Q}, \beta > t > s} \mathcal{G}_{S \vee t}) \\
&= \{S \vee \beta < T\} \cap \mathcal{G}_s^{(S,T]}
\end{aligned}
$$

Consider then a set $B \in \{S \vee s < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]})$. For any $\beta > s$,

$$B \cap \{S \vee \beta < T\} \in \{S \vee \beta < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) = \{S \vee \beta < T\} \cap \mathcal{G}_s^{(S,T]}$$

Let $B_\beta \in \mathcal{G}_s^{(S,T]}$ such that $B \cap \{S \vee \beta < T\} = B_\beta \cap \{S \vee \beta < T\}$. We have $B_\beta \cap \{S \vee \beta < T\} = B_{\beta'} \cap \{S \vee \beta < T\}$ for any $\beta > \beta' > s$. Let $B_* = \cap_{\epsilon > 0} \cup_{\beta' \in \mathbb{Q}, s < \beta' < s + \epsilon} B_{\beta'} \in \mathcal{G}_s^{(S,T]}$. We have

$$
\begin{aligned}
B_* \cap \{S \vee \beta < T\} &= \{S \vee \beta < T\} \cap (\cap_{\epsilon > 0} \cup_{s < \beta' < s + \epsilon} B_{\beta'}) \\
&= \{S \vee \beta < T\} \cap (\cap_{\beta - s > \epsilon > 0} \cup_{s < \beta' < s + \epsilon} B_{\beta'}) \\
&= \cap_{\beta - s > \epsilon > 0} \cup_{s < \beta' < s + \epsilon} (\{S \vee \beta < T\} \cap B_{\beta'}) \\
&= \cap_{\beta - s > \epsilon > 0} \cup_{s < \beta' < s + \epsilon} (\{S \vee \beta < T\} \cap B) \\
&= B \cap \{S \vee \beta < T\}
\end{aligned}
$$

Taking the union on $\beta > s$ we obtain finally

$$B = B \cap \{S \vee s < T\} = B_* \cap \{S \vee s < T\}$$

This being true for any $B \in \{S \vee s < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]})$, we obtain

$$\{S \vee s < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) \subset \{S \vee s < T\} \cap \mathcal{G}_s^{(S,T]}$$

Actually we have an equality instead of the inclusion, because the inverse of the above relation is an evidence. Since $\{S \vee s < T\} \in \mathcal{G}_s^{(S,T]}$, we can put together the two equalities :

$$
\begin{aligned}
\{T \leq S \vee s\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) &= \{T \leq S \vee s\} \cap \mathcal{G}_s^{(S,T]} \\
\{S \vee s < T\} \cap (\cap_{t > s} \mathcal{G}_t^{(S,T]}) &= \{S \vee s < T\} \cap \mathcal{G}_s^{(S,T]}
\end{aligned}
$$

and we conclude $\mathcal{G}_s^{(S,T]} = \cap_{t > s} \mathcal{G}_t^{(S,T]}$, i.e., the right continuity of $\mathbb{G}^{(S,T]}$. ∎

Lemma A.5 S and T are $\mathbb{G}^{(S,T]}$-stopping times. More generally, any $\mathbb{G}$-stopping time R between S and $T : S \leq R \leq T$, is a $\mathbb{G}^{(S,T]}$-stopping time. Conversely, for any $\mathbb{G}^{(S,T]}$-stopping time R', $S \vee R'$ is a $\mathbb{G}$-stopping time.

This lemma can be checked straightforwardly.

Lemma A.6 *(Proposition 4.1 (2)) For a $\mathbb{G}^{(S,T]}$-stopping time R such that $S \leq R \leq T$, we have*

$$\mathcal{G}_R^{(S,T]} = \{R = T\} \cap \mathcal{G}_T^* + \{R < T\} \cap \mathcal{G}_R = \{R = T\} \cap \mathcal{G}_R^* + \{R < T\} \cap \mathcal{G}_R$$

In particular, $\mathcal{G}_S^{(S,T]} = \mathcal{G}_0^{(S,T]}$ and $\mathcal{G}_T^{(S,T]} = \mathcal{G}_T^*$.

Proof. Consider first $\mathcal{G}_T^{(S,T]}$. We have, on the one hand, $\mathcal{G}_T^{(S,T]} \subset \mathcal{G}_\infty^{(S,T]} = \mathcal{G}_T^*$. On the other hand, for any $B \in \mathcal{G}_T^*$, for $t \geq 0$, we note that $\{T \leq t\} \in \mathcal{G}_{T-} \subset \mathcal{G}_T^*$ so that $B \cap \{T \leq t\} \in \mathcal{G}_T^*$. Therefore,

$$
\begin{aligned}
B \cap \{T \leq t\} &= B \cap \{T \leq t\} \cap \{T \leq S \vee t\} + B \cap \{T \leq t\} \cap \{S \vee t < T\} \\
&= B \cap \{T \leq t\} \cap \{T \leq S \vee t\} \\
&\in \{T \leq S \vee t\} \cap \mathcal{G}_T^* \\
&\in \mathcal{G}_t^{(S,T]}
\end{aligned}
$$

This proves that $B \in \mathcal{G}_T^{(S,T]}$, i.e., $\mathcal{G}_T^{(S,T]} = \mathcal{G}_T^*$.

Consider now $\mathcal{G}_R^{(S,T]}$. We note that R is also a $\mathbb{G}$-stopping time, as it is stated in the preceding Lemma A.5. Let $A \in \mathcal{G}_R^{(S,T]}$. Then, for any $t \geq 0$, since $\{R < T\}$ is an element in $\mathcal{G}_R^{(S,T]}$, we have

$$
\begin{aligned}
A \cap \{R < T\} \cap \{R \leq t\} &\in \{R \leq t\} \cap \mathcal{G}_t^{(S,T]} \\
&\subset \{R \leq t\} \cap \mathcal{G}_{(S \vee t) \wedge T} \quad \text{according to Lemma A.3} \\
&\subset \{R \leq t\} \cap \{S \leq t\} \cap \mathcal{G}_{S \vee t} \\
&\subset \mathcal{G}_t
\end{aligned}
$$

This computation shows that $A \cap \{R < T\} \in \mathcal{G}_R$. Conversely let $B \in \mathcal{G}_R$. For $t \geq 0$, we have

$$B \cap \{R < T\} \cap \{R \leq t\} \in \mathcal{G}_{T-} \text{ and } B \cap \{R < T\} \cap \{R \leq t\} \in \mathcal{G}_t,$$

Therefore,

$$
\begin{aligned}
&B \cap \{R < T\} \cap \{R \leq t\} \\
={}& \{T \leq S \vee t\} \cap B \cap \{R < T\} \cap \{R \leq t\} + \{S \vee t < T\} \cap B \cap \{R < T\} \cap \{R \leq t\} \\
\in{}& \{T \leq S \vee t\} \cap \mathcal{G}_{T-} + \{S \vee t < T\} \cap \{R \leq t\} \cap \mathcal{G}_t \\
\subset{}& \{T \leq S \vee t\} \cap \mathcal{G}_T^* + \{S \vee t < T\} \cap \mathcal{G}_{S \vee t} \\
={}& \mathcal{G}_t^{(S,T]}
\end{aligned}
$$

This proves that $B \cap \{R < T\} \in \mathcal{G}_R^{(S,T]}$. We have just established

$$\{R < T\} \cap \mathcal{G}_R^{(S,T]} = \{R < T\} \cap \mathcal{G}_R$$

We can also write

$$\{R = T\} \cap \mathcal{G}_R^{(S,T]} = \{R = T\} \cap \mathcal{G}_T^{(S,T]} = \{R = T\} \cap \mathcal{G}_T^*$$

(For the first equality, see [13, Corollary 3.5 statement 4)]). The lemma is proved. ∎

Lemma A.7 *(Proposition 4.1 (3)) For any $\mathbb{G}$-adapted process X such that $X_T \in \mathcal{G}_T^*$, $X^{(S,T]}$ defines a $\mathbb{G}^{(S,T]}$-adapted process. Conversely, for any $\mathbb{G}^{(S,T]}$-adapted process X', $X'^{(S,T]}$ defines a $\mathbb{G}$-adapted process.*

This lemma can be checked using Lemma A.3.

Lemma A.8 *(Proposition 4.1 (5)) For any* $(\mathbb{Q}, \mathbb{G})$ *local martingale* X *such that* $X_T \in \mathcal{G}_T^*$, $X^{(S,T]}$ *defines a* $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ *local martingale. Conversely, for any* $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ *local martingale* X', $X'^{(S,T]}$ *defines a* $(\mathbb{Q}, \mathbb{G})$ *local martingale.*

Proof. Let R be a X-reducing stopping time, i.e., a $\mathbb{G}$-stopping time such that X^R is a uniformly $\mathbb{Q}$-integrable $(\mathbb{Q}, \mathbb{G})$ martingale. Note that, for $t \geq 0$, $(X^{(S,T]})_t^R = (X^R)_t^{(S,T]}$. As $\mathbb{1}_{\{R<T\}}X_R$, $\mathbb{1}_{\{R \geq T\}} \in \mathcal{G}_{T-}$, we have

$$X_T^R = \mathbb{1}_{\{R<T\}}X_R + \mathbb{1}_{\{R \geq T\}}X_T \in \mathcal{G}_T^*$$

This relation together with the preceding Lemma A.7 shows that $(X^{(S,T]})^R$ is $\mathbb{G}^{(S,T]}$-adapted. (We note that we could not say this directly, because R is not a $\mathbb{G}^{(S,T]}$-stopping time.)

Let $0 \leq s \leq t \leq \infty$ and $A \in \mathcal{G}_s^{(S,T]}$. Since $\mathcal{G}_s^{(S,T]} \subset \mathcal{G}_{(S \vee s) \wedge T} \subset \mathcal{G}_{(S \vee s)}$, we have

$$\mathbb{E}[\mathbb{1}_A(X^R)_{S \vee t}^T] = \mathbb{E}[\mathbb{1}_A X_{S \vee t}^{R \wedge T}] = \mathbb{E}[\mathbb{1}_A X_{S \vee s}^{R \wedge T}] = \mathbb{E}[\mathbb{1}_A(X^R)_{S \vee s}^T]$$

This computation implies

$$\mathbb{E}[\mathbb{1}_A(X^R)_{S \vee t}^{(S,T]}] = \mathbb{E}[\mathbb{1}_A(X^R)_{S \vee s}^{(S,T]}]$$

i.e. $(X^R)^{(S,T]} = (X^{(S,T]})^R$ is a $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ uniformly integrable martingale. We note that $X_t^{(S,T]} = 0$ if $t \leq S$ and $X_t^{(S,T]} = X_T - X_S$ if $t \geq T$. We can write

$$(X^{(S,T]})^R = (X^{(S,T]})^{R'}$$

where

$$R' \quad = \quad \mathbb{1}_{\{(S \vee R) \wedge T < T\}}(S \vee R) \wedge T + \mathbb{1}_{\{(S \vee R) \wedge T \geq T\}} \cdot \infty$$

According to Lemma A.5, $(S \vee R) \wedge T$, as well as T, is a $\mathbb{G}^{(S,T]}$-stopping time. The set $\{(S \vee R) \wedge T < T\}$ is in $\mathcal{G}_{(S \vee R) \wedge T}^{(S,T]}$. Hence, R', as a restriction of $(S \vee R) \wedge T$, is also a $\mathbb{G}^{(S,T]}$-stopping time. Moreover, if R tends to infinity, R' does so too. We can now state that $X^{(S,T]}$ is a $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ local martingale. The first part of the lemma is proved.

Consider a $(\mathbb{Q}, \mathbb{G}^{(S,T]})$ local martingale X'. Let U' be a X'-reducing stopping time. We know that $X'^{(S,T]}$ is a $\mathbb{G}$-adapted process, and $S \vee U'$ is a $\mathbb{G}$-stopping time. We use the identity

$$(X'^{(S,T]})_t^{S \vee U'} \quad = \quad (X'^{S \vee U'})_t^{(S,T]} = (X'^{S \vee U'})_{S \vee t}^T - X_S'^{S \vee U'} = X_{S \vee t}^{'(S \vee U') \wedge T} - X_S'$$

for $t \geq 0$. Set $R = (S \vee U') \wedge T$ which is a stopping time with respect to both $\mathbb{G}$ and $\mathbb{G}^{(S,T]}$. Let $0 \leq s \leq t \leq \infty$ and $B \in \mathcal{G}_s$. We write

$$\mathbb{E}[\mathbb{1}_B(X'^{(S,T]})_t^{S \vee U'}] \quad = \quad \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{T \leq S \vee s\}}(X_{S \vee t}'^R - X_S')] + \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}}(X_{S \vee t}'^R - X_S')]$$

We note that $B \cap \{S \vee s < T\} \in \mathcal{G}_{S \vee s}$ which yields $B \cap \{S \vee s < T\} \in \mathcal{G}_s^{(S,T]}$. Hence,

$$\mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}}(X_{S \vee t}'^R - X_S')] \quad = \quad \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}} \mathbb{1}_{\{S < U'\}}(X_{S \vee t}'^{U' \wedge T} - X_S')]$$
$$= \quad \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}} \mathbb{1}_{\{S < U'\}}(X_{S \vee s}'^{U' \wedge T} - X_S')]$$
$$\text{because } \{S < U'\} \in \mathcal{G}_{S \vee s}^{(S,T]}$$
$$= \quad \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}}(X_{S \vee s}'^R - X_S')]$$

We note also

$$\mathbb{1}_B \mathbb{1}_{\{T \leq S \vee s\}}(X_{S \vee t}'^R - X_S') = \mathbb{1}_B \mathbb{1}_{\{T \leq S \vee s\}}(X_R' - X_S') = \mathbb{1}_B \mathbb{1}_{\{T \leq S \vee s\}}(X_{S \vee s}'^R - X_S')$$

Putting these relations together, we write

$$\mathbb{E}[\mathbb{1}_B(X'^{(S,T]})_t^{S \vee U'}] \quad = \quad \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{T \leq S \vee s\}}(X_{S \vee s}'^R - X_S')] + \mathbb{E}[\mathbb{1}_B \mathbb{1}_{\{S \vee s < T\}}(X_{S \vee s}'^R - X_S')]$$
$$= \quad \mathbb{E}[\mathbb{1}_B(X_{S \vee s}'^R - X_S')]$$
$$= \quad \mathbb{E}[\mathbb{1}_B(X'^{(S,T]})_t^{S \vee U'}]$$

This proves the second part of the lemma. ∎

Lemma A.9 *(Proposition 4.1 (4)) For any $\mathbb{G}$-predictable process K, the processes $K^{(S,T]}$ and $\mathbb{1}_{(S,T]}K$ are $\mathbb{G}^{(S,T]}$-predictable. Conversely, for any $\mathbb{G}^{(S,T]}$-predictable process K', $K'^{(S,T]}$ and $\mathbb{1}_{(S,T]}K'$ are $\mathbb{G}$-predictable processes.*

Proof. If K or K' are left continuous, with Lemma A.7, the result is clear. To pass to general K or K', we use the monotone class theorem. ∎

Lemma A.10 *For any increasing process A, $(\mathbb{1}_{(S,T]}).A$ is $(\mathbb{Q},\mathbb{G})$-locally integrable if and only if $(\mathbb{1}_{(S,T]}).A$ is $(\mathbb{Q},\mathbb{G}^{(S,T]})$-locally integrable. In the case of a locally integrable A, let A'' and A' be respectively the $(\mathbb{Q},\mathbb{G})$ and the $(\mathbb{Q},\mathbb{G}^{(S,T]})$ dual predictable projection of $(\mathbb{1}_{(S,T]}).A$. Then, we have $(\mathbb{1}_{(S,T]}).A'' = (\mathbb{1}_{(S,T]}).A'$*

This lemma can be checked straightforwardly.

Lemma A.11 *(Proposition 4.1 (6)) For any $\mathbb{G}$-predictable process K, for any $(\mathbb{Q},\mathbb{G})$ local martingale X such that $X_T \in \mathcal{G}_T^*$, the fact that $\mathbb{1}_{(S,T]}K$ is X-integrable in $\mathbb{G}$ implies that $\mathbb{1}_{(S,T]}K$ is $X^{(S,T]}$-integrable in $\mathbb{G}^{(S,T]}$. Conversely, for any $\mathbb{G}^{(S,T]}$-predictable process K', for any $(\mathbb{Q},\mathbb{G}^{(S,T]})$ local martingale X', the fact that $\mathbb{1}_{(S,T]}K'$ is X'-integrable in $\mathbb{G}^{(S,T]}$ implies that $\mathbb{1}_{(S,T]}K'$ is $X'^{(S,T]}$-integrable in $\mathbb{G}$.*

Proof. We note that the brackets $[X^{(S,T]}]$ and $[X'^{(S,T]}]$ are the same in the two filtrations. We can then check that $\sqrt{K.[X^{(S,T]}]}$ or $\sqrt{K.[X'^{(S,T]}]}$ is locally integrable in one filtration if and only if it is so in the other filtration. ∎

We end the Appendix by the following result

Lemma A.12 *(Lemma 4.1) Let S,T be two $\mathbb{G}$-stopping times. We have*

$$\mathcal{G}_{S\vee T}^* = \{S < T\} \cap \mathcal{G}_T^* + \{T \leq S\} \cap \mathcal{G}_S^*$$

Proof. Let us consider only this identity on the set $\{S < T\} \cap \{\tau \leq S \vee T < \infty\}$. With Lemma A.1 we can write

$$
\begin{aligned}
&\{S < T\} \cap \{\tau \leq S \vee T < \infty\} \cap \mathcal{G}_{S\vee T}^* \\
=\ &\{S < T\} \cap \{\tau \leq T < \infty\} \cap \sigma\{\{\tau \leq s\}, \{X_{S\vee T} \leq t\}:\ X \text{ is } \mathbb{F} \text{ optional}, s,t \in \mathbb{R}\} \\
=\ &\{S < T\} \cap \{\tau \leq T < \infty\} \cap \sigma\{\{\tau \leq s\}, \{X_T \leq t\}:\ X \text{ is } \mathbb{F} \text{ optional}, s,t \in \mathbb{R}\} \\
=\ &\{S < T\} \cap \{\tau \leq T < \infty\} \cap \mathcal{G}_T^*
\end{aligned}
$$

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